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Topological cyclic homology of the integers at two

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Abstract

The topological Hochschild homology of the integers $T(\mathbb{Z}) = THH(\mathbb{Z})$ is an S^1-equivariant spectrum. We prove by computation that for the restricted C_2-action on $T(\mathbb{Z})$ the fixed points and homotopy fixed points are equivalent, after passing to connective covers and completing at two. By Tsalidis (1994) a similar statement then holds for the action of every cyclic subgroup $C_{2^n} \subset S^1$ of order a power of two. Next we inductively determine the mod two homotopy groups of all the fixed point spectra $T(\mathbb{Z})^{C_{2^n}}$, following Bökstedt and Madsen (1994, 1995) and Tsalidis (1994). We also compute the restriction maps relating these spectra, and use this to find the mod two homotopy groups of the topological cyclic homology of the integers $TC(\mathbb{Z})$, and of the algebraic K-theory of the two-adic integers $K(\hat{\mathbb{Z}}_2)$. © 1999 Elsevier Science B.V. All rights reserved.

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0. Introduction

In this paper we compute the mod two homotopy groups of the topological cyclic homology of the integers.

Let A be a ring. Fix a prime p and let all spaces, spectra and homotopy groups be implicitly completed at p. The *topological Hochschild homology* of A is an S^1-equivariant spectrum $T(A) = THH(A)$. The *topological cyclic homology* of A is a spectrum $TC(A)$ defined as a suitable homotopy limit of the fixed point subspectra $T(A)^{C_{p^n}}$, for varying n. Here $C_{p^n} \subset S^1$ denotes the cyclic subgroup with p^n elements. There is

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a *cyclotomic trace map*

$$\mathrm{trc} : K(A) \to TC(A)$$

from the *algebraic K-theory spectrum* $K(A)$ of A to its topological cyclic homology. See [5] or [11] for these constructions. In this paper we consider the basic case $A = \mathbb{Z}$ and $p = 2$. We compute the mod two homotopy groups of the fixed point subspectra $T(\mathbb{Z})^{C_{2^n}}$ for all n, and use this to determine the mod two homotopy groups of $TC(\mathbb{Z})$.

Let us first outline the main line of argument.

To study the fixed point spectra $T(A)^{C_{p^n}}$ we make a comparison with the homotopy fixed point spectra $T(A)^{hC_{p^n}}$. There is a natural *comparison map*

$$\Gamma_n : T(A)^{C_{p^n}} \to T(A)^{hC_{p^n}}$$

which in favorable circumstances induces an isomorphism on mod p homotopy in non-negative degrees. Then Γ_n induces a homotopy equivalence of p-adically completed connective covers. We call such maps *connective p-adic equivalences*. The fixed point spectra $T(A)^{C_{p^n}}$ are always connective.

There is also a spectral sequence, with E^2-term

$$(0.1) \qquad E^2_{s,t} = H^{-s}(C_{p^n}, \pi_t(T(A); \mathbb{Z}/p))$$

abutting to $\pi_{s+t}(T(A)^{hC_{p^n}}; \mathbb{Z}/p)$. In the first part of this paper (Sections 1–4) we use this spectral sequence (with p=2 and n=1) to compute the mod two homotopy of $T(\mathbb{Z})^{hC_2}$. We also show that the comparison map $\Gamma_1 : T(\mathbb{Z})^{C_2} \to T(\mathbb{Z})^{hC_2}$ is a connective two-adic equivalence. Hence we obtain a calculation of the mod two homotopy of $T(\mathbb{Z})^{C_2}$.

Let us write $X[0,\infty)$ for the connective cover of a spectrum X, and X_p for its p-adic completion.

Theorem 0.2. *The map Γ_1 is a connective two-adic equivalence. Hence the induced map*

$$\Gamma_1 : T(\mathbb{Z})_2^{C_2} \to T(\mathbb{Z})_2^{hC_2}[0,\infty)$$

is a homotopy equivalence.

We prove this result as Theorem 4.7. This provides the calculational input for the following theorem of Tsalidis, proved in [20].

Theorem 0.3 (Tsalidis [20]). *Fix a prime p. If the comparison map Γ_1 is a connective p-adic equivalence, then so is Γ_n for all $n \geq 1$.*

Hence $T(\mathbb{Z})^{C_{2^n}}$ is two-adically equivalent to the connective cover of $T(\mathbb{Z})^{hC_{2^n}}$ for all $n \geq 1$, and the spectral sequence (0.1) can be used for $p = 2$ and all $n \geq 1$ to compute the mod two homotopy of $T(\mathbb{Z})^{C_{2^n}}$. This is carried out in the second part of this paper (Sections 5–10).

The detailed structure of these spectral sequences is determined in Section 8. See Lemma 8.4 and Theorem 8.14. The resulting mod two homotopy calculations are described in Section 9. These lead to a calculation of the mod two homotopy of $TC(\mathbb{Z})$.

Theorem 0.4. *The mod two homotopy groups of $TC(\mathbb{Z})$ have orders*

$$\#\pi_*(TC(\mathbb{Z});\mathbb{Z}/2)=\begin{cases} 2 & \text{for } *=-1,0,\\ 4 & \text{for } *\geq 2 \text{ even or } *=1,\\ 8 & \text{for } *\geq 3 \text{ odd.}\end{cases}$$

The remaining groups are trivial.

We prove this as Theorem 10.9. Section 10 also gives more precise statements about the generators for the groups $\pi_*(TC(\mathbb{Z});\mathbb{Z}/2)=TC_*(\mathbb{Z};\mathbb{Z}/2)$ and their multiplicative relations.

The main interest in topological cyclic homology stems from its close relationship to algebraic K-theory. The following result is a special case of Theorem D of [11], combining a result of McCarthy on relative K-theory with continuity results of Hesselholt and Madsen. Let $\hat{\mathbb{Z}}_p$ be the ring of p-adic integers.

Theorem 0.5 (McCarthy, Hesselholt–Madsen). *The cyclotomic trace map*

$$\mathrm{trc}: K(\hat{\mathbb{Z}}_p)\to TC(\hat{\mathbb{Z}}_p)\cong TC(\mathbb{Z})$$

is a connective p-adic equivalence.

Hence the calculation of $TC(\mathbb{Z})$ amounts to a p-adic calculation of $K(\hat{\mathbb{Z}}_p)$. For odd primes p this was carried out by Bökstedt and Madsen in [6, 7]. They obtained a p-adic homotopy equivalence of infinite loop spaces

$$K(\hat{\mathbb{Z}}_p)_p\simeq J_p\times BJ_p\times BBU_p.$$

Here J_p is the p-primary *image of J-space*, BJ_p its first delooping, and $BBU_p\simeq SU_p$ the p-completed infinite special unitary group.

For $p=2$ the results of the present paper will be used in [18] to give a two-adic calculation of $K(\hat{\mathbb{Z}}_2)$ – both in terms of giving its homotopy groups, and by expressing it as an infinite loop space. The answer at $p=2$ is built from the same components as for p odd, but J_2 must be interpreted as the *complex* image of J-space, and the product splitting above is replaced by two nontrivial fibrations.

We next review the construction of the cyclotomic trace map in a little more detail, to fix notations. See [11] for the full discussion.

The fixed point spectra $T(A)^{C_{p^n}}$ are related by natural *restriction* and *Frobenius* maps

$$R,F: T(A)^{C_{p^n}}\to T(A)^{C_{p^{n-1}}},$$

denoted Φ and D respectively in [5].

The Dennis–Bökstedt trace map $\mathrm{tr}: K(A) \to T(A)$ admits lifts $\mathrm{tr}_{p^n}: K(A) \to T(A)^{C_{p^n}}$, which are compatible with R and F up to chosen homotopies. Let $\mathbb{I}_p$ be the following category:

$$1 \underset{F}{\overset{R}{\leftrightarrows}} p \underset{F}{\overset{R}{\leftrightarrows}} \cdots \underset{F}{\overset{R}{\leftrightarrows}} p^{n-1} \underset{F}{\overset{R}{\leftrightarrows}} p^n \underset{F}{\overset{R}{\leftrightarrows}} \cdots$$

By definition $TC(A)_p = \mathrm{holim}_{p^n \in \mathbb{I}_p} T(A)_p^{C_{p^n}}$. The lifts tr_{p^n} and chosen homotopies define the cyclotomic trace map $\mathrm{trc}: K(A)_p \to TC(A)_p$, which is well defined up to homotopy after p-adic completion. We let $TF(A)_p = \mathrm{holim}_F T(A)_p^{C_{p^n}}$. There is then a natural fiber sequence

$$TC(A)_p \xrightarrow{\pi} TF(A)_p \xrightarrow{R-1} TF(A)_p.$$

As an application of Theorems 0.2 and 0.3 we can recognize the intermediate trace invariant $TF(\mathbb{Z})_2$ as something more familiar.

Corollary 0.6. *There are homotopy equivalences*

$$TF(\mathbb{Z})_2 = \operatorname*{holim}_F T(\mathbb{Z})_2^{C_{2^n}} \xrightarrow{\simeq} \operatorname*{holim}_F T(\mathbb{Z})_2^{hC_{2^n}}[0,\infty) \xleftarrow{\simeq} T(\mathbb{Z})_2^{hS^1}[0,\infty).$$

In particular the cyclotomic trace map lifts the circle trace map $\mathrm{tr}_{S^1}: K(A)_p \to T(A)_p^{hS^1}$, which was discussed in [16].

$$\begin{tikzcd}
K(\hat{\mathbb{Z}}_2)_2 \arrow[d, "\mathrm{trc}"'] \arrow[dr, "\mathrm{tr}_{S^1}"] & & \\
TC(\mathbb{Z})_2 \arrow[r, "\pi"] & TF(\mathbb{Z})_2 \arrow[r, "R-1"] & TF(\mathbb{Z})_2.
\end{tikzcd}$$

Hence there are exact sequences

$$0 \to K_{2r+1}(\hat{\mathbb{Z}}_2)_2 \xrightarrow{\pi} \pi_{2r+1} T(\mathbb{Z})_2^{hS^1} \xrightarrow{R-1} \pi_{2r+1} T(\mathbb{Z})_2^{hS^1} \xrightarrow{\partial} K_{2r}(\hat{\mathbb{Z}}_2)_2 \to 0$$

for all $r > 0$.

We now outline the various sections of this paper. The first part of the paper consists of Sections 1 to 4. Its main aim is to prove Theorem 0.2.

In Section 1 we review and extend Bökstedt's two-primary analysis of the circle trace map $\mathrm{tr}_{S^1}: K(\mathbb{Z}) \to T(\mathbb{Z})^{hS^1}$, starting with the results from [16]. Because there is no (natural) algebra structure on the mod two homotopy of a ring spectrum, we are led to consider the action of its mod four homotopy upon its mod two homotopy, studied by Oka in [15]. In particular we need the action of the mod four homotopy of $T(\mathbb{Z})$ upon its mod two homotopy, discussed in [17]. This constitutes an added complication compared to the odd primary case, when mod p homotopy admits natural products.

In Section 2 we recall from [10] the *Tate construction* $\hat{\mathbb{H}}(G,T(\mathbb{Z}))$ for $G\subseteq S^1$, and set up the spectral sequences abutting to the mod two or mod four homotopy of the homotopy fixed point spectra $T(\mathbb{Z})^{hC_{2^n}}$ and the Tate constructions $\hat{\mathbb{H}}(C_{2^n},T(\mathbb{Z}))$. There is a fundamental map of horizontal fiber sequences:

$$\text{(0.7)}\qquad \begin{array}{ccccc} T(\mathbb{Z})_{hC_{2^n}} & \xrightarrow{N_n} & T(\mathbb{Z})^{C_{2^n}} & \xrightarrow{R_n} & T(\mathbb{Z})^{C_{2^{n-1}}} \\ \| & & \downarrow{\scriptstyle \Gamma_n} & & \downarrow{\scriptstyle \hat{\Gamma}_n} \\ T(\mathbb{Z})_{hC_{2^n}} & \xrightarrow{N_n^h} & T(\mathbb{Z})^{hC_{2^n}} & \xrightarrow{R_n^h} & \hat{\mathbb{H}}(C_{2^n},T(\mathbb{Z})) \end{array}$$

We use this diagram for $n=1$ to determine the first differentials in the spectral sequence (0.1). This approach also gives a simple proof of Theorem 5.8(i) of [6], for any p.

In Section 3 we consider a double ladder of maps relating the mod two and mod four spectral sequences for various groups C_{2^n} to one another. The maps are induced by Frobenius and *Verschiebung* maps, parallel to restriction and transfer maps in group cohomology. Naturality considerations among these spectral sequences place strong restrictions on where their first differentials of *odd* length may appear.

In Section 4 we compute the spectral sequences for the mod two homotopy of the C_2-homotopy fixed points and the C_2-Tate construction on $T(\mathbb{Z})$. The absence of a natural algebra structure on the mod two spectral sequence is replaced by a study of the action of the mod four spectral sequence upon the mod two spectral sequence. It is therefore necessary to make a partial calculation of the mod four homotopy spectral sequence as well, coupled with the mod two computation.

Considering diagram (0.7) it is clear that Γ_1 is a connective two-adic equivalence if and only if the related map $\hat{\Gamma}_1 : T(\mathbb{Z})\to\hat{\mathbb{H}}(C_2,T(\mathbb{Z}))$ is a connective two-adic equivalence. This is what we prove in Theorem 4.7. The mod two homotopy groups of both sides are known by the spectral sequence calculations, and the map is shown to induce an isomorphism in nonnegative degrees by a comparison with $T(\mathbb{F}_2)$, for which the result is known from [11]. Conversely, diagram (0.7) shows that by Tsalidis' Theorem 0.3, each map $\hat{\Gamma}_n : T(\mathbb{Z})^{C_{2^{n-1}}}\to\hat{\mathbb{H}}(C_{2^n},T(\mathbb{Z}))$ is a connective two-adic equivalence for all $n\geq 1$.

The second part of the paper consists of Sections 5–10. These constitute an inductive argument along the lines of [20], computing the mod two homotopy of $T(\mathbb{Z})^{hC_{2^n}}$ and $\hat{\mathbb{H}}(C_{2^n},T(\mathbb{Z}))$ from the mod two homotopy of the corresponding spectra involving $C_{2^{n-1}}$.

In Section 5 we discuss short exact sequences of spectral sequences. We give a criterion in Proposition 5.4 for when a diagram of three spectral sequences that forms a short exact sequence at the E^1-term, persists to give short exact sequences of E^r-terms for all $r\geq 1$.

In Section 6 we apply this to the diagram of three spectral sequences computing the homotopy of $\hat{\mathbb{H}}(S^1, T(\mathbb{Z}))$ smashed with either of the three spectra in the following cofiber sequence:

$$M \xrightarrow{i} M \wedge M \xrightarrow{j} \Sigma M.$$

Here $M = S^0/2$ is the mod two Moore spectrum. We use this in Proposition 6.5 to internalize an external product on the spectral sequence $\hat{E}^*(S^1; \mathbb{Z}/2)$ computing $\pi_*(\hat{\mathbb{H}}(S^1, T(\mathbb{Z})); \mathbb{Z}/2)$, and show that its differentials are derivations. This result is particular to $A = \mathbb{Z}$, and may not hold for general rings. As an application we relate the even (resp. odd) columns of the spectral sequence computing $\pi_*(\hat{\mathbb{H}}(C_{2^n}, T(\mathbb{Z})); \mathbb{Z}/2)$ to the even columns of $\hat{E}^*(S^1; \mathbb{Z}/2)$, in a stable range.

In Sections 7 and 8 we inductively determine the spectral sequences computing the mod two homotopy of $T(\mathbb{Z})^{C_{2^n}}$. Section 7 illustrates the step from $n=1$ to $n=2$; the latter section covers the general case. Theorem 8.14 gives the complete answer. Our argument largely follows the ideas of [6, 20], but we are also able to make some simplifications. For instance our characterizations in Lemmas 9.3 and 9.7 substitute for the p-series from Section 4 of [6].

Here is how the proof can be thought of as an inductive argument. The inductive hypothesis (8.1) for n assumes complete knowledge of the upper half plane spectral sequence $\hat{E}^*(C_{2^n}; \mathbb{Z}/2)$ computing the mod two homotopy of the Tate construction $\hat{\mathbb{H}}(C_{2^n}, T(\mathbb{Z}))$. By restriction to the second quadrant, this determines the spectral sequence $E^*(C_{2^n}; \mathbb{Z}/2)$ computing the mod two homotopy of the homotopy fixed points $T(\mathbb{Z})^{hC_{2^n}}$. In non-negative degrees this agrees via the comparison map Γ_n with the mod two homotopy of the fixed points $T(\mathbb{Z})^{C_{2^n}}$ (using Theorems 0.2 and 0.3), which in turn agrees via the other comparison map $\hat{\Gamma}_{n+1}$ with the mod two homotopy of the next Tate construction $\hat{\mathbb{H}}(C_{2^{n+1}}, T(\mathbb{Z}))$. See diagram (0.7). This gives the abutment of the spectral sequence $\hat{E}^*(C_{2^{n+1}}; \mathbb{Z}/2)$, and the remaining work is to recover the pattern of differentials in the spectral sequence leading up to this abutment. This uses the fragments of multiplicative structure available, and recovers the inductive hypothesis for $n+1$.

In Sections 9 and 10 we analyze the answer, providing the information needed to assemble $TC(\mathbb{Z})$ from the fixed points $T(\mathbb{Z})^{C_{2^n}}$. In Section 9 we characterize the permanent cycles in the spectral sequence computing the mod two homotopy of $T(\mathbb{Z})^{C_{2^n}}$, for each n. See Definition 9.2 and Lemmas 9.3 and 9.4. In the final section we determine the restriction maps $R: T(\mathbb{Z})^{C_{2^n}} \to T(\mathbb{Z})^{C_{2^{n-1}}}$ on mod two homotopy, which suffices to determine the mod two homotopy of $TC(\mathbb{Z})$. This is the main output or result of the present paper, given in Theorem 10.9. Equivalently, this computes the mod two homotopy of $K(\hat{\mathbb{Z}}_2)$.

In the sequel [18] to this paper, we will construct maps relating $K(\hat{\mathbb{Z}}_2)$ to known spaces (like the image of J-spaces mentioned at the beginning), so as to fiber off known parts from $K(\hat{\mathbb{Z}}_2)$, until the remaining piece is characterized (by its mod two homotopy and being nearly K-local) as being equivalent to $BBU \simeq SU$. The fiber sequences involved will be classified by means of the multiplicative structures available, and a complete description of the infinite loop space $K(\hat{\mathbb{Z}}_2)_2$ results. Finally the natural map $K(\mathbb{Z}) \to K(\hat{\mathbb{Z}}_2)$ is studied, providing an interesting invariant of the algebraic K-theory of the integers.

For convenience, let us gather together the definitions of various classes in the homotopy of $Q(S^0)$, $K(\mathbb{Z})$ and $K(\hat{\mathbb{Z}}_2)$ which will be defined in the course of the paper. All names are carried over along the natural ring maps $Q(S^0) \to K(\mathbb{Z}) \to K(\hat{\mathbb{Z}}_2)$ without further mention in the notation.

Notation 0.8. Let η, v and σ denote generators of $\pi_1 Q(S^0)_2 \cong \mathbb{Z}/2$, $\pi_3 Q(S^0)_2 \cong \mathbb{Z}/8$ and $\pi_7 Q(S^0)_2 \cong \mathbb{Z}/16$, respectively. Then $\eta^3 = 4v$. Let λ denote a generator of $K_3(\mathbb{Z})_2 \cong \mathbb{Z}/16$, such that $2\lambda = v$. Then $2\eta = 0$ so there is a class $\tilde{\eta}_2 \in \pi_2(Q(S^0); \mathbb{Z}/2)$ with mod two Bockstein $j_1(\tilde{\eta}_2) = \eta$. Furthermore $4v = \eta^3 = 0$ in $K_3(\hat{\mathbb{Z}}_2)$, so there is a class $\tilde{v}_4 \in K_4(\hat{\mathbb{Z}}_2; \mathbb{Z}/4)$ with mod four Bockstein $j_2(\tilde{v}_4) = v$. Its mod two reduction is denoted $\rho\tilde{v}_4 \in K_4(\hat{\mathbb{Z}}_2; \mathbb{Z}/2)$. There is a class $\kappa \in K_5(\mathbb{Z})_2$ with mod two reduction $i_1(\kappa) = \lambda\tilde{\eta}_2$ since $\lambda\eta = 0$ in $K_4(\mathbb{Z})$. It generates $K_5(\mathbb{Z})_2$ modulo torsion, which is $\hat{\mathbb{Z}}_2$. Finally there is a class $\bar{\sigma} \in K_7(\hat{\mathbb{Z}}_2)_2$ with mod four reduction $i_2(\bar{\sigma}) = \lambda\tilde{v}_4$, since $\lambda v = 2\lambda^2 = 0$. We shall prove in [18] that σ and $\bar{\sigma}$ agree in $K_7(\hat{\mathbb{Z}}_2)_2$.

1. The circle trace map

We begin by recalling Bökstedt's analysis of the circle trace map $\mathrm{tr}_{S^1} : K(\hat{\mathbb{Z}}_2)_2 \to T(\mathbb{Z})_2^{hS^1} = \mathrm{Map}(ES^1_+, T(\mathbb{Z}))_2^{S^1}$. The skeleton filtration on the simplicial space ES^1_+ gives a homological upper left quadrant algebra spectral sequence

$$E^2_{s,*}(S^1) = H^{-s}(S^1; T_*(\mathbb{Z})_2) \cong \begin{cases} T_*(\mathbb{Z})_2 & \text{for } s \le 0 \text{ even,} \\ 0 & \text{otherwise,} \end{cases} \tag{1.1}$$

converging to $\pi_{s+*} T(\mathbb{Z})_2^{hS^1}$. Here $H^*(S^1; M)$ refers to group cohomology with coefficients in a discrete module M. There can be no group action on M, because S^1 is path connected.

Recall from [4] that the nonzero homotopy groups of $T(\mathbb{Z})_2$ are $T_0(\mathbb{Z})_2 \cong \hat{\mathbb{Z}}_2$ and $T_{2i-1}(\mathbb{Z})_2 \cong \mathbb{Z}/i \otimes \hat{\mathbb{Z}}_2 \cong \mathbb{Z}/2^{v_2(i)}$ for $i > 0$ even. Here $v_2(i)$ is the two-adic valuation of i. We choose additive generators $g_{4k-1} \in T_{4k-1}(\mathbb{Z})_2$ of order $2^{v_2(k)+1}$.

Let $\mathscr{A} = H^{spec}_*(H\mathbb{Z}/2; \mathbb{Z}/2) \cong \mathbb{Z}/2[\xi_1, \xi_2, \xi_3, \dots]$ be the dual of the mod two Steenrod algebra, and let $\overline{\mathscr{A}} = H^{spec}_*(H\mathbb{Z}; \mathbb{Z}/2) \cong \mathbb{Z}/2[\xi_1^2, \chi\xi_2, \chi\xi_3, \dots]$ where χ is the canonical involution. Then $H^{spec}_*(T(\mathbb{Z}); \mathbb{Z}/2) \cong \overline{\mathscr{A}}[e_3, e_4]/(e_3^2 = 0)$ as $\overline{\mathscr{A}}$-algebras, with $e_n \in H^{spec}_n(T(\mathbb{Z}); \mathbb{Z}/2)$ for $n = 3, 4$. The spherical class g_{4k-1} maps to $e_3 e_4^{k-1}$ under the Hurewicz homomorphism and mod two reduction. See [17] for further discussion.

Also recall that $H^*(S^1; \mathbb{Z}) \cong \mathbb{Z}[t]$, and more generally $H^*(S^1; M) \cong M[t]$ for every module M, where $t \in H^2(S^1; \mathbb{Z})$ is a fixed generator. Thus the E^2-term of the spectral sequence (1.1) appears as depicted in Fig. 1.2 below, with the origin in the bottom

$\mathbb{Z}/8$		$\mathbb{Z}/8$		$\mathbb{Z}/8$		$\mathbb{Z}/8$		$\mathbb{Z}/8$	g_{15}
$2\bar{\sigma}$									
$\mathbb{Z}/2$		$\mathbb{Z}/2$		$\mathbb{Z}/2$		$\mathbb{Z}/2$		$\mathbb{Z}/2$	g_{11}
		4κ		$\bar{\sigma}$					
$\mathbb{Z}/4$		$\mathbb{Z}/4$		$\mathbb{Z}/4$		$\mathbb{Z}/4$		$\mathbb{Z}/4$	g_7
				ν		κ			
$\mathbb{Z}/2$		$\mathbb{Z}/2$		$\mathbb{Z}/2$		$\mathbb{Z}/2$		$\mathbb{Z}/2$	g_3
						η		λ	
$\mathbb{Z}$		$\mathbb{Z}$		$\mathbb{Z}$		$\mathbb{Z}$		$\mathbb{Z}$	1
t^4		t^3		t^2		t		1	

Fig. 1.2. $E^*(S^1) \Rightarrow \pi_* T(\mathbb{Z})_2^{hS^1}$.

right corner. The empty bidegrees contain trivial groups. The labels $\eta, \lambda, \ldots, 2\bar{\sigma}$ indicate classes mapping to the generator of the group directly above the label.

Note that all classes above the horizontal axis sit in odd total degrees. Thus $\pi_* T(\mathbb{Z})^{hS^1}$ for $* > 0$ is concentrated in odd degrees. In particular there are no differentials originating above the horizontal axis.

Consider the natural maps of infinite loop spaces

$$(1.3) \qquad Q(S^0) \to K(\mathbb{Z}) \to K(\hat{\mathbb{Z}}_2) \xrightarrow{\mathrm{tr}_{S^1}} T(\mathbb{Z})_2^{hS^1}.$$

Classes in $\pi_* Q(S^0)$, $K_*(\mathbb{Z})$ and $K_*(\hat{\mathbb{Z}}_2)$ that map nontrivially to $\pi_* T(\mathbb{Z})_2^{hS^1}$ survive as permanent cycles in the spectral sequence.

The infinite loop spaces in (1.3) are all E_∞ ring spaces, but it remains to be proved that the cyclotomic trace map $\mathrm{trc}: K(A)_p \to TC(A)_p$ is multiplicative, and similarly for $\mathrm{tr}_{S^1}: K(A)_p \to T(\mathbb{Z})_p^{hS^1}$. Hence we will not assume that the circle trace map above induces algebra homomorphisms in homotopy. On the other hand, the trace map $\mathrm{tr}: K(A) \to T(A)$ is a map of E_∞ ring spaces, e.g., by the construction using hyper-Γ-spaces in Section 2 of [3].

Notation 1.4. Let $\eta \in \pi_1 Q(S^0) \cong \mathbb{Z}/2$ be the class of the complex Hopf map $S^3 \to S^2$. Then η^2 generates $\pi_2 Q(S^0) \cong \mathbb{Z}/2$, and η^3 is a class of order two in $\pi_3 Q(S^0)$. Let

$\nu \in \pi_3 Q(S^0)_2$ be the class of the quaternionic Hopf map $S^7 \to S^4$. Then ν generates $\pi_3 Q(S^0)_2 \cong \mathbb{Z}/8$, and $4\nu = \eta^3$.

Choose a generator $\lambda \in K_3(\mathbb{Z})_2 \cong \mathbb{Z}/16$ (see [12]) such that $2\lambda = \nu$, where we identify $\nu \in \pi_3 Q(S^0)_2$ with its image in $K_3(\mathbb{Z})_2$. Clearly $8\lambda = \eta^3$.

Theorem 1.5. (1) *The image of* η *survives as* $tg_3 \in E^{\infty}_{-2,3}$ *in the spectral sequence* (1.1).
(2) *The image of* λ *survives as* $g_3 \in E^{\infty}_{0,3}$.
(3) *The classes* η^2 *and* η^3 *map to zero in* $\pi_* T(\mathbb{Z})_2^{hS^1}$.
(4) *The product* $\eta\lambda$ *is zero in* $K_4(\mathbb{Z})_2$.
(5) *The product* η^3 *is zero in* $K_3(\hat{\mathbb{Z}}_2)_2$.

Proof. The first two claims are due to Marcel Bökstedt. See [16] for proofs. The next claim is clear, since $\pi_* T(\mathbb{Z})_2^{hS^1}$ is concentrated in odd degrees and the maps (1.3) are module maps over $\pi_* Q(S^0)$.

For the fourth claim, we use the splitting of two-completed looped underlying spaces $\phi : \Omega JK(\mathbb{Z})_2 \hookrightarrow \Omega K(\mathbb{Z})_2$ of [2]. Here $JK(\mathbb{Z})_2$ is the homotopy fiber of the composite

$$ko_2 \xrightarrow{\psi^3 - 1} bspin_2 \xrightarrow{c} bsu_2,$$

which comes equipped with a natural four-connected map $\Phi : K(\mathbb{Z})_2 \to JK(\mathbb{Z})_2$. Then $\Omega\Phi \circ \phi \simeq 1$ on $\Omega JK(\mathbb{Z})_2$, and $\lambda = \phi(\lambda') \in K_3(\mathbb{Z})_2$ with $\lambda' = \Phi(\lambda)$, so $\eta\lambda$ factors on the space level as

$$S^3 \xrightarrow{\eta} S^2 \xrightarrow{\lambda'} \Omega JK(\mathbb{Z})_2 \hookrightarrow \Omega K(\mathbb{Z})_2.$$

Hence $\eta\lambda$ is trivial as $\eta\lambda' \in JK_4(\mathbb{Z})_2 = 0$. An alternative proof is given by Arlettaz in [1].

I have the following direct proof of claim (5) from C.A. Weibel. Let $\hat{\mathbb{Q}}_2 = \hat{\mathbb{Z}}_2[\frac{1}{2}]$ be the two-adic numbers. It suffices to prove that $\eta^3 = 0$ in $K_3(\hat{\mathbb{Q}}_2)$, since $K_3(\hat{\mathbb{Z}}_2)_2 \to K_3(\hat{\mathbb{Q}}_2)_2$ is an isomorphism by the localization sequence. We compute with the multi-multiplicative symbols $\{a_1, \ldots, a_n\}$ in Milnor K-theory $K_n^M(\hat{\mathbb{Q}}_2)$, where the a_i's are units in $\hat{\mathbb{Q}}_2$. We write Milnor K-theory multiplicatively, so $\{a_1, a_2\} = 1$ if $a_1 + a_2 = 1$. Then the symbol $\{-1\}$ represents the image of η in $K_1^M(\hat{\mathbb{Q}}_2) = K_1(\hat{\mathbb{Q}}_2)$, and we need to prove that $\{-1, -1, -1\} = 1$ (since $K_n^M(\hat{\mathbb{Q}}_2)$ maps to $K_n(\hat{\mathbb{Q}}_2)$ for all n).

The quadratic residue symbol $K_2^M(\hat{\mathbb{Q}}_2) = K_2(\hat{\mathbb{Q}}_2) \to \{\pm 1\} \cong \mathbb{Z}/2$ is also called the Hilbert symbol $(a,b)_2$, and detects $K_2(\mathbb{Z})$ in $K_2(\hat{\mathbb{Q}}_2)$. By definition $(a,b)_2 = +1$ if the equation $ax^2 + by^2 = 1$ has any solutions with $x, y \in \hat{\mathbb{Q}}_2$, and otherwise $(a,b)_2 = -1$. Because $\hat{\mathbb{Q}}_2$ is a local field, the kernel of the quadratic residue symbol map $\{a,b\} \mapsto (a,b)_2$ is a divisible group by a theorem of Calvin Moore. See Chapter 11 and Theorem A.14 of [14] for more on this.

In particular $(-1,-1)_2 = (2,5)_2 = -1$, so $\{-1,-1\}\{2,5\}^{-1}$ is in the divisible group, and can therefore be written as x^2 for some $x \in K_2^M(\hat{\mathbb{Q}}_2)$. Thus $\{-1,-1\} = \{2,5\}x^2$.

Also $\{-1,2\}=1$ since $-1+2=1$, and $\{-1,x^2\}=\{(-1)^2,x\}=1$ by bi-multiplicativity. Thus

$$\{-1,-1,-1\}=\{-1,2,5\}\cdot\{-1,x^2\}=1\cdot 1=1$$

is trivial in $K_3^M(\hat{\mathbb{Q}}_2)$. $\square$

Next we wish to work with homotopy with finite coefficients. See [17] for a fuller discussion. Let

$$S^0 \xrightarrow{2^v} S^0 \xrightarrow{i_v} S^0/2^v \xrightarrow{j_v} S^1$$

be the Puppe cofibration sequence of spectra induced by the degree 2^v map $S^0\to S^0$. This defines the mod 2^v Moore spectrum $S^0/2^v$. The mod 2^v homotopy of a spectrum X is defined as $\pi_*(X;\mathbb{Z}/2^v)=\pi_*(X\wedge S^0/2^v)$. We write $K_*(A;\mathbb{Z}/2^v)=\pi_*(K(A);\mathbb{Z}/2^v)$ and $T_*(A;\mathbb{Z}/2^v)=\pi_*(T(A);\mathbb{Z}/2^v)$ when $v\geq 1$. There is a product map $\mu_2:S^0/4\wedge S^0/4\to S^0/4$, and a module pairing $m:S^0/4\wedge S^0/2\to S^0/2$, but no unital product on $S^0/2$. See [15].

There is a coefficient reduction map $\rho:S^0/4\to S^0/2$ with $\rho\circ i_2\simeq i_1$ and $j_1\circ\rho\simeq 2\circ j_2$. There is a coefficient extension map $\varepsilon:S^0/2\to S^0/4$ with $\varepsilon\circ i_1\simeq 2\circ i_2$ and $j_2\circ\varepsilon\simeq j_1$. Let $\delta_v=i_v\circ j_v:S^0/2^v\to S^1/2^v$ be the homotopy Bockstein map. These maps fit into a cofiber sequence of spectra

$$S^0/2 \xrightarrow{\varepsilon} S^0/4 \xrightarrow{\rho} S^0/2 \xrightarrow{\delta_1} S^1/2.$$

The product μ_2 is regular, in the sense that when X is a ring spectrum δ_2 acts as a derivation on $\pi_*(X;\mathbb{Z}/4)$. Hereafter we shall often omit the unit maps i_v from the notation.

$T(\mathbb{Z})$ is a homotopy commutative ring spectrum, but for degree reasons the product on $T_*(\mathbb{Z})$ is trivial in positive degrees. However the product map on $S^0/4$ induces a nontrivial commutative $\mathbb{Z}/4$-algebra structure on $T_*(\mathbb{Z};\mathbb{Z}/4)$, and a nontrivial module pairing of $T_*(\mathbb{Z};\mathbb{Z}/4)$ upon $T_*(\mathbb{Z};\mathbb{Z}/2)$. The following theorem was proved in [17].

Theorem 1.6. (1) *The mod two spherical elements*

$$T_*(\mathbb{Z};\mathbb{Z}/2)=\pi_*(T(\mathbb{Z});\mathbb{Z}/2)\subset H_*^{spec}(T(\mathbb{Z});\mathbb{Z}/2)\cong\overline{\mathscr{A}}[e_3,e_4]/(e_3^2=0)$$

are closed under the algebra product, so form a subalgebra $T_*(\mathbb{Z};\mathbb{Z}/2)\cong\mathbb{Z}/2[e_3,e_4]/(e_3^2=0)$.

(2) *We can choose generators* $f_n\in T_n(\mathbb{Z};\mathbb{Z}/4)$ *for* $n=3,4,7$ *and* 8, *so that* $\delta_2(f_4)=f_3$, $\delta_2(f_8)=f_7$ *and*

$$T_*(\mathbb{Z};\mathbb{Z}/4)\cong\mathbb{Z}/4[f_3,f_4,f_7,f_8]/\\(2f_3=2f_4=0,\ f_if_j=0\ \text{for } i,\ j<8,\ \text{except } f_3f_4=2f_7)$$

as algebras.

(3) *The coefficient reduction map* $\rho: T_*(\mathbb{Z};\mathbb{Z}/4) \to T_*(\mathbb{Z};\mathbb{Z}/2)$ *is an algebra map given by* $\rho(f_3)=e_3$, $\rho(f_4)=0$, $\rho(f_7)=e_3e_4$ *and* $\rho(f_8)=e_4^2$.

(4) *The module action of* $T_*(\mathbb{Z};\mathbb{Z}/4)$ *on* $T_*(\mathbb{Z};\mathbb{Z}/2)$ *is given by the coefficient reduction map* $\rho: T_*(\mathbb{Z};\mathbb{Z}/4) \to T_*(\mathbb{Z};\mathbb{Z}/2)$ *followed by the subalgebra product on* $T_*(\mathbb{Z};\mathbb{Z}/2)$.

(5) *The coefficient extension map* $\varepsilon: T_*(\mathbb{Z};\mathbb{Z}/2) \to T_*(\mathbb{Z};\mathbb{Z}/4)$ *is a* $T_*(\mathbb{Z};\mathbb{Z}/4)$*-module map given by* $\varepsilon(1)=2$, $\varepsilon(e_3)=0$, $\varepsilon(e_4)=f_4$ *and* $\varepsilon(e_3e_4)=2f_7$.

Proof. See Theorem 3.2 of [17]. □

There is a universal coefficient short exact sequence

$$0 \to \pi_*(X)/2^v \xrightarrow{i_v} \pi_*(X;\mathbb{Z}/2^v) \xrightarrow{j_v} {}_{2^v}\pi_{*-1}(X) \to 0$$

which is split for $v \geq 2$, but not necessarily split for $v=1$. The sequence is split also for $v=1$ if multiplication by η induces a trivial map $\eta: {}_2\pi_{*-1}(X) \to \pi_*(X)/2$. For example this is the case with $X = T(\mathbb{Z})_2^{hS^1}[0,\infty)$, since $\pi_0(X) \cong \mathbb{Z}$ is torsion free and $\pi_*(X)$ is concentrated in odd degrees for $* > 0$. Hence $\pi_*(T(\mathbb{Z})_2^{hS^1};\mathbb{Z}/2)$ has exponent two in positive degrees.

By analogy with the spectral sequence (1.1), there are spectral sequences arising from the skeleton filtration on EG_+

$$(1.7) \qquad E^2_{s,*}(G;\Lambda) = H^{-s}(G; T_*(\mathbb{Z};\Lambda)) \Rightarrow \pi_{s+*}(T(\mathbb{Z})^{hG};\Lambda)$$

for every (closed) subgroup $G \subseteq S^1$ and coefficient ring $\Lambda = \hat{\mathbb{Z}}_2$ or $\Lambda = \mathbb{Z}/2^v$ with $v \geq 1$. We write $E^*(G) = E^*(G;\hat{\mathbb{Z}}_2)$, in agreement with the case $G=S^1$. The other closed subgroups of S^1 are the cyclic subgroups C_q. Recall that $H^0(C_{2^n};M) \cong M$ when M is a trivial C_{2^n}-module, while we can identify $H^k(C_{2^n};M) \cong {}_{2^n}M$ for $k>0$ odd and $H^k(C_{2^n};M) \cong M/2^n$ for $k>0$ even. Then $H^2(C_{2^n};\mathbb{Z}) \cong \mathbb{Z}/2^n$ is generated by the restriction of $t \in H^2(S^1;\mathbb{Z})$, which we also denote by t. Let $u_n \in H^1(C_{2^n};\mathbb{Z}/2) \cong \mathbb{Z}/2$ and $u_n' \in H^1(C_{2^n};\mathbb{Z}/4) \cong {}_{2^n}\mathbb{Z}/4$ be fixed generators for every $n \geq 1$. The u_n and u_n' can and will be chosen to be compatible under the group transfer and coefficient extension maps.

The spectral sequences $E^*(G)$ and $E^*(G;\mathbb{Z}/2^v)$ are algebra spectral sequences for $v \geq 2$, with product on the E^2-term induced by the algebra structure on $T_*(\mathbb{Z})_2$ or $T_*(\mathbb{Z};\mathbb{Z}/2^v)$, and the cup product on cohomology. Similarly there is a natural module action of the spectral sequence $E^*(G;\mathbb{Z}/4)$ upon $E^*(G;\mathbb{Z}/2)$. However, as we shall see, $E^*(G;\mathbb{Z}/2)$ is *not* generally an algebra spectral sequence, when we give the E^2-term the algebra structure induced by the subalgebra product on $T_*(\mathbb{Z};\mathbb{Z}/2)$ and the cohomology cup product. With $\mathbb{Z}/2$-coefficients the E^2-terms have the following form:

$$E^2_{s,*}(C_{2^n};\mathbb{Z}/2) \cong \begin{cases} \mathbb{Z}/2 & \text{when } s \leq 0 \text{ and } * \equiv 0,3 \bmod 4 \text{ is nonnegative,} \\ 0 & \text{otherwise,} \end{cases}$$

so

$$E^2_{*,*}(C_{2^n};\mathbb{Z}/2)\cong \mathbb{Z}/2[t,u_n,e_3,e_4]/(u_n^2=0,e_3^2=0)$$

for $n\geq 2$, replacing $u_1^2=0$ with $u_1^2=t$ when $n=1$. Here $u_n\in E^2_{-1,0}$, $t\in E^2_{-2,0}$, $e_3\in E^2_{0,3}$ and $e_4\in E^2_{0,4}$. The algebra structure given is induced by the subalgebra structure on $T_*(\mathbb{Z};\mathbb{Z}/2)$ inherited from $H^{spec}_*(T(\mathbb{Z});\mathbb{Z}/2)$, and since this might not be compatible with the S^1-action on $T(\mathbb{Z})$ we cannot a priori conclude that the d^r-differentials in these spectral sequences will act as derivations. However it will be convenient to express the behavior of the spectral sequences in terms of these algebra structures on the above E^2-terms.

With $\mathbb{Z}/4$-coefficients we have the following E^2-term:

$$\begin{aligned} E^2_{*,*}(S^1;\mathbb{Z}/4) &\cong T_*(\mathbb{Z};\mathbb{Z}/4)[t] \\ &\cong \mathbb{Z}/4[t,f_3,f_4,f_7,f_8]/\sim, \end{aligned}$$

where $\sim$ denotes the relations of Theorem 1.6(2), while $E^2_{*,*}(C_{2^n};\mathbb{Z}/4)$ is a little more complicated to write down. (But see Lemma 2.5 and the formulas after Lemma 3.2 below.) Here $t\in E^2_{-2,0}$, and $f_k\in E^2_{0,k}$. These are algebra spectral sequences, by naturality of the product on mod four homotopy. Unlike the mostly trivial algebra structure on the integral spectral sequences $E^*(G)$, the algebra structure on $E^*(G;\mathbb{Z}/4)$ and $E^2(G;\mathbb{Z}/2)$ is certainly nontrivial.

The algebra structure on $E^*(G;\mathbb{Z}/4)$ is commutative, because the commutator factors through the action of η^2, which maps to zero in $E^*(S^1)$ by Theorem 1.5(3), and thus in all the $E^*(G;A)$ we consider. See Section 1 of [17] regarding this commutator. In fact $T(\mathbb{Z})^{hS^1}\wedge S^0/4$ will be a commutative ring spectrum, and similarly for $T(\mathbb{Z})^{hC_{2^n}}\wedge S^0/4$ and $\hat{\mathbb{H}}(C_{2^n},T(\mathbb{Z}))\wedge S^0/4$ (defined in Section 2), since the commutator map factors through a map induced by smashing with $\eta^2:S^2\to S^0$, which is inessential on all module spectra of $T(\mathbb{Z})^{hS^1}_2$ by Theorem 1.5(3).

Classes in $\pi_*(Q(S^0);\mathbb{Z}/2^v)$, $K_*(\mathbb{Z};\mathbb{Z}/2^v)$ and $K_*(\hat{\mathbb{Z}}_2;\mathbb{Z}/2^v)$ map under (1.3) to infinite cycles in the spectral sequence $E^*(S^1;\mathbb{Z}/2^v)$. We will now describe these maps in low degrees, for $v=1$ and 2.

Notation 1.8. Choose classes $\tilde{\eta}_2\in\pi_2(Q(S^0);\mathbb{Z}/2)$ and $\tilde{\eta}_4\in\pi_2(Q(S^0);\mathbb{Z}/4)$ so that $j_1(\tilde{\eta}_2)=\eta$ and $j_2(\tilde{\eta}_4)=\eta$. Choose classes $\tilde{\lambda}_8\in K_4(\hat{\mathbb{Z}}_2;\mathbb{Z}/8)$, $\tilde{v}_4\in K_4(\hat{\mathbb{Z}}_2;\mathbb{Z}/4)$ and $\widetilde{2v}_2\in K_4(\hat{\mathbb{Z}}_2;\mathbb{Z}/2)$ so that $j_3(\tilde{\lambda}_8)=\lambda$, $j_2(\tilde{v}_4)=v$ and $j_1(\widetilde{2v}_2)=2v$. This is possible because $8\lambda=4v=2\cdot 2v=\eta^3=0$ in $K_3(\hat{\mathbb{Z}}_2)_2$, by Theorem 1.5(5) above. Our convention is to let $\tilde{x}_{2^v}$ denote a class with $j_v(\tilde{x}_{2^v})=x$.

With the right sign choices for these lifts the following theorem holds:

Theorem 1.9. (1) *$\tilde{\eta}_2$ maps to $te_4\in E^\infty_{-2,4}(S^1;\mathbb{Z}/2)$ and $\tilde{\eta}_4$ maps to $tf_4\in E^\infty_{-2,4}(S^1;\mathbb{Z}/4)$.*

(2) *There are surjective differentials $d^4(t)=t^3e_3$ in $E^*(S^1;\mathbb{Z}/2)$, $d^4(t)=t^3f_3$ in $E^*(S^1;\mathbb{Z}/4)$ and $d^4(t)=t^3g_3$ in $E^*(S^1)$.*

(3) *There is a nontrivial differential* $d^4(e_4)=t^2e_3e_4$ *in* $E^*(S^1;\mathbb{Z}/2)$, *while* $d^4(f_4)=2t^2f_7$ *in* $E^*(S^1;\mathbb{Z}/4)$. *Hence there is a nontrivial extension*

$$\mathbb{Z}/4=E^{\infty}_{-4,7}\to\mathbb{Z}/8\to\mathbb{Z}/2=E^{\infty}_{0,3}$$

in $E^*(S^1)$.

(4) $v=2\lambda$ *maps to* $t^2g_7\in E^{\infty}_{-4,7}(S^1)$, *to* $t^2e_3e_4\in E^{\infty}_{-4,7}(S^1;\mathbb{Z}/2)$ *and to* $t^2f_7\in E^{\infty}_{-4,7}(S^1;\mathbb{Z}/4)$.

(5) $\widetilde{2v_2}$ *maps to* $t^2e_4^2\in E^{\infty}_{-4,8}(S^1;\mathbb{Z}/2)$ *and* $\tilde v_4$ *maps to* $t^2f_8\in E^{\infty}_{-4,8}(S^1;\mathbb{Z}/4)$.

We give a partial proof now, postponing the essential claim that $d^4(t)=t^3g_3$ to Proposition 2.7.

Proof. (1) The skeleton filtration on ES^1 agrees with the filtration $S^1\subset S^3\subset\cdots\subset S^\infty$ by the unit spheres $S(\mathbb{C}^n)\subset\mathbb{C}^n$ of $S(\mathbb{C}^\infty)\simeq ES^1$. In particular the inclusion $S^3_+\to ES^1_+$ induces a map

$$T(\mathbb{Z})_2^{hS^1}\to \mathrm{Map}(S^3_+,T(\mathbb{Z}))_2^{S^1}=X,$$

which on the level of spectral sequences arising from the skeleton filtration above induces the truncation of $E^*(S^1;\Lambda)$ to its two rightmost nonzero columns $s=-2$ and $s=0$. See Section 3 of [16] for a discussion of this map.

By Theorem 1.5(1) $\eta\in\pi_1Q(S^0)$ maps to tg_3 in the resulting two-column spectral sequence for π_*X. So $\tilde\eta_2\in\pi_2(Q(S^0);\mathbb{Z}/2)$ must map to a class $x\in\pi_2(X;\mathbb{Z}/2)$ with $j_1(x)$ represented by tg_3. Thus x is nonzero, and the only nonzero class in total degree two of $E^*_{s,*}(S^1;\mathbb{Z}/2)$ with $s=-2$ or $s=0$ is $te_4\in E^*_{-2,4}(S^1;\mathbb{Z}/2)$. So $\tilde\eta_2$ maps to te_4.

Similarly $\tilde\eta_4\in\pi_2(Q(S^0);\mathbb{Z}/4)$ must map to a class y in $\pi_2(X;\mathbb{Z}/4)$ with $j_2(y)$ represented by tg_3, and the only nonzero candidate is tf_4.

(2) We postpone the calculation of the differential $d^4(t)=t^3g_3$ in $E^*(S^1)$ until we have introduced the Tate construction and the norm-restriction fiber sequence in Section 2.

Assuming this, the claims $d^4(t)=t^3e_3$ in $E^*(S^1;\mathbb{Z}/2)$ and $d^4(t)=t^3f_3$ in $E^*(S^1;\mathbb{Z}/4)$ follow by naturality with respect to the coefficient reduction maps $i_v:S^0\to S^0/2^v$ for $v=1,2$.

(3) tf_4 is the image of $\tilde\eta_4$ and therefore an infinite cycle in $E^*(S^1;\mathbb{Z}/4)$. Hence

$$0=d^4(tf_4)=d^4(t)f_4+td^4(f_4)=t^3f_3f_4+td^4(f_4)$$

by (2). Since $f_3f_4=2f_7$ by Theorem 1.6(2), and multiplication by t is injective in $E^4(S^1;\mathbb{Z}/4)$, we obtain $d^4(f_4)=2t^2f_7$.

By naturality with respect to the coefficient extension map $\varepsilon:S^0/2\to S^0/4$, with $\varepsilon(e_4)=f_4$ and $\varepsilon(e_3e_4)=2f_7$ by Theorem 1.6(5), we find $d^4(e_4)=t^2e_3e_4$.

By considering the three-column spectral sequence with $s=-4$, $s=-2$ or $s=0$ for $\pi_*(Y;\Lambda)$ with $Y=\mathrm{Map}(S^5_+,T(\mathbb{Z}))_2^{S^1}$, and the universal coefficient short exact sequence, it is clear that the differential $d^4(e_4)=t^2e_3e_4$ in $E^*(S^1;\mathbb{Z}/2)$ corresponds to a nontrivial extension in $\pi_3 Y$, and thus also in $E^*(S^1)$. For only $\mathbb{Z}/2\{e_3\}$ survives in total degree 3 in the truncated spectral sequence for $\pi_*(Y;\mathbb{Z}/2)$, so $(\pi_3 Y)/2 \cong \mathbb{Z}/2$ and $\pi_3 Y$ must be cyclic.

(4) The nontrivial extension from (3) asserts that twice the generator λ of $\pi_3 Y$ must map to the generator of $\mathbb{Z}/4 \cong E^{\infty}_{-4,7}(S^1)$, i.e., that v maps to $t^2 g_7$. The claims for mod two and mod four homotopy follow by coefficient reduction.

(5) v maps to $t^2 e_3 e_4$ in the three-column spectral sequence for $\pi_*(Y;\mathbb{Z}/2)$. Hence $\widetilde{2v_2} \in K_4(\hat{\mathbb{Z}}_2;\mathbb{Z}/2)$ must map to a class $z \in \pi_4(Y;\mathbb{Z}/2)$ with $\delta_1(z)$ represented by $t^2 e_3 e_4$. So z is nonzero, and the only nonzero class in total degree four of $E^{\infty}_{s,*}(S^1;\mathbb{Z}/2)$ with $-4 \le s \le 0$ is $t^2 e_4^2 \in E^{\infty}_{-4,8}(S^1;\mathbb{Z}/2)$. Thus $\widetilde{2v_2}$ maps to $t^2 e_4^2$.

Similarly $\tilde{v}_4 \in K_4(\hat{\mathbb{Z}}_2;\mathbb{Z}/4)$ must map to a class $w \in \pi_4(Y;\mathbb{Z}/4)$ with $\delta_2(w)$ represented by $t^2 f_7$. Now $\delta_2(f_8) = f_7$, so the only possibility is $\tilde{v}_4 \mapsto t^2 f_8$. $\square$

Remark 1.10. Apparently it is necessary to use mod four homotopy with its algebra structure, rather than just integral and mod two homotopy and the pairing between them, in order to translate the differential $d^4(t) = t^3 e_3$ into the nontrivial extension in total degree three.

We proceed to define certain classes in $K_*(\mathbb{Z})$ and $K_*(\hat{\mathbb{Z}}_2)$, which are detected in $\pi_* T(\mathbb{Z})_2^{hS^1}$.

Lemma 1.11. (1) *There is a nonzero class $\kappa \in K_5(\mathbb{Z})$ defined modulo 2 by $i_1(\kappa) = \lambda\tilde{\eta}_2$, which maps to $tg_7 \in E^{\infty}_{-2,7}(S^1)$.*

(2) *There is a nonzero class $\bar{\sigma} \in K_7(\hat{\mathbb{Z}}_2)$ defined modulo 4 by $i_2(\bar{\sigma}) = \lambda\tilde{v}_4$, which maps to $t^2 g_{11} \in E^{\infty}_{-4,11}(S^1)$.*

Proof. By Theorem 1.6(4), $j_1(\lambda\tilde{\eta}_2) = \eta\lambda = 0$, so $\kappa \in K_5(\mathbb{Z})$ is uniquely determined modulo 2. Since $\lambda \mapsto g_3$ integrally, and $\tilde{\eta}_2 \mapsto te_4$ mod two, we get that κ maps to a class in $E^*(S^1)$ which gives $g_3 \cdot te_4 = te_3 e_4$ under mod two reduction, i.e., $\pm tg_7 \in E^{\infty}_{-2,7}(S^1)$. We can choose the plus sign.

(The remainder of the proof is complicated by the fact that we do not know whether tr_{S^1} is multiplicative. Instead we use that it is a module map over $\pi_* Q(S^0)$.)

We find $j_2(\lambda\tilde{v}_4) = \lambda v = 2\lambda^2 = 0$ since $K_*(\hat{\mathbb{Z}}_2)$ is a graded commutative algebra and λ is of odd degree. So $\bar{\sigma} \in K_7(\hat{\mathbb{Z}}_2)$ is well defined modulo 4 by $i_2(\bar{\sigma}) = \lambda\tilde{v}_4$. Then $i_2(2\bar{\sigma}) = v\tilde{v}_4$ and so $i_2(\mathrm{tr}_{S^1}(2\bar{\sigma})) = \mathrm{tr}_{S^1}(v \cdot \tilde{v}_4) = v \cdot \mathrm{tr}_{S^1}(\tilde{v}_4)$, which maps to $t^2 g_7 \cdot t^2 f_8 = t^4 f_7 f_8$ in $E^*(S^1;\mathbb{Z}/4)$.

Since $\mathrm{tr}: K(\hat{\mathbb{Z}}_2) \to T(\hat{\mathbb{Z}}_2) \cong T(\mathbb{Z})_2$ is multiplicative we get $i_2(\mathrm{tr}(\bar{\sigma})) = g_3 \cdot 0 = 0$, so $\mathrm{tr}_{S^1}(\bar{\sigma})$ is not detected in $E^2_{0,7}(S^1)$ and must have filtration ≤ -4. The group $E^2_{-4,11}(S^1) = \mathbb{Z}/2$, so $\mathrm{tr}_{S^1}(2\bar{\sigma})$ must have filtration ≤ -8. Since its mod four reduction maps to $t^4 f_7 f_8$ be the calculation above, it must have filtration precisely -8. So $\mathrm{tr}_{S^1}(2\bar{\sigma})$ is represented by an odd multiple of $t^4 g_{15} \ne 0$, which is not divisible by 2 in $E^2_{-8,15}(S^1)$. Hence $\mathrm{tr}_{S^1}(\bar{\sigma})$ is represented by $t^2 g_{11}$, as claimed. $\square$

When we have determined the first additive extensions in $E^*(S^1)$, in Corollary 4.4 below, the following additional classes can be detected.

Lemma 1.12. *$\kappa \in K_5(\mathbb{Z})$ has order at least eight, and $4\kappa \mapsto t^3 g_{11}$ in $E^{\infty}_{-6,11}(S^1)$. $\bar{\sigma} \in K_7(\hat{\mathbb{Z}}_2)$ has order at least 16, and $2\bar{\sigma} \mapsto t^4 g_{15}$ in $E^{\infty}_{-8,15}(S^1)$ (up to an odd multiple).*

Proof. We assume Corollary 4.4. Then $K_5(\mathbb{Z}) \to \pi_5 T(\mathbb{Z})_2^{hS^1} \twoheadrightarrow \mathbb{Z}/8 \twoheadrightarrow \mathbb{Z}/4$ takes κ to a generator of $\mathbb{Z}/4$, whence κ generates a cyclic group of order at least eight. The argument in degree seven is similar, after fibering over $T(\mathbb{Z})_2$ to avoid the class g_7. □

Remark 1.13. The class κ represents a generator in $K_5(\mathbb{Z})$ modulo torsion, which is $\mathbb{Z}$ by [8]. Hence κ can be chosen to generate a direct $\mathbb{Z}$-summand in $K_5(\mathbb{Z})$. To prove this, use Bökstedt's map $\Phi : K(\mathbb{Z})_2 \to JK(\mathbb{Z})_2$ from [2], which maps $i_1(\kappa) = \lambda\tilde{\eta}_2$ to the generator $\lambda'\tilde{\eta}_2$ for $\pi_5(JK(\mathbb{Z}); \mathbb{Z}/2)$, and so takes κ to a generator of $JK_5(\mathbb{Z})_2 = \hat{\mathbb{Z}}_2$. Hence $\pi_5(\Phi)$ identifies $K_5(\mathbb{Z})_2$ modulo its torsion subgroup with $JK_5(\mathbb{Z})_2 = \hat{\mathbb{Z}}_2$. Presumably there is no torsion in $K_5(\mathbb{Z})$.

We will prove in [18] that $\bar{\sigma}$ and σ agree mod two in $K_7(\hat{\mathbb{Z}}_2)_2$. If $\kappa\eta = 0$ in $K_6(\mathbb{Z})$ it still remains to prove that $i_1(\sigma)$ represents $\kappa\tilde{\eta}_2$ in $K_7(\mathbb{Z}; \mathbb{Z}/2)$.

2. The Tate construction on $T(\mathbb{Z})$

Let G be a compact Lie group, and T a G-spectrum indexed on a complete G-universe, in the sense of [13]. There is a G-cofibration sequence of G-spaces

$$EG_+ \to S^0 \to \tilde{E}G$$

defining $\tilde{E}G$, where c collapses EG to a point. The Tate construction for G acting on T is defined in [10] to be

$$\hat{\mathbb{H}}(G, T) = [\tilde{E}G \wedge \mathrm{Map}(EG_+, T)]^G.$$

This is the G-fixed point spectrum of the Tate spectrum denoted $t_G(T)$ in loc.cit. Smashing the G-cofibration sequence above with the adjoint $c : T \to \mathrm{Map}(EG_+, T)$, and taking G-fixed point spectra, we obtain the following map of fiber sequences

$$\begin{tikzcd}
{[EG_+ \wedge T]^G} \arrow[r] \arrow[d] & T^G \arrow[r] \arrow[d] & {[\tilde{E}G \wedge T]^G} \arrow[d] \\
{[EG_+ \wedge \mathrm{Map}(EG_+, T)]^G} \arrow[r] & \mathrm{Map}(EG_+, T)^G \arrow[r] & {[\tilde{E}G \wedge \mathrm{Map}(EG_+, T)]^G}.
\end{tikzcd}$$

Now suppose $G=C_{2^n}\subset S^1$ and $T=T(\mathbb{Z})$. Then since $T(\mathbb{Z})$ is a cyclotomic spectrum, by Section 4 of [11] the diagram above is homotopy equivalent to the following diagram:

$$\begin{tikzcd} T(\mathbb{Z})_{hC_{2^n}} \arrow[r, "N"] \arrow[d, equal] & T(\mathbb{Z})^{C_{2^n}} \arrow[r, "R"] \arrow[d, "\Gamma_n"] & T(\mathbb{Z})^{C_{2^{n-1}}} \arrow[d, "\hat{\Gamma}_n"] \\ T(\mathbb{Z})_{hC_{2^n}} \arrow[r, "N^h"] & T(\mathbb{Z})^{hC_{2^n}} \arrow[r, "R^h"] & \hat{\mathbb{H}}(C_{2^n}, T(\mathbb{Z})) \end{tikzcd} \tag{2.1}$$

We call the top fiber sequence the *norm-restriction sequence*.

The lifted trace maps $\mathrm{tr}_{2^n}:K(\hat{\mathbb{Z}}_2)\to T(\mathbb{Z})_2^{C_{2^n}}$ are compatible up to homotopy under the subspace inclusions $F:T(\mathbb{Z})_2^{C_{2^n}}\subset T(\mathbb{Z})_2^{C_{2^{n-1}}}$. Furthermore, by Proposition 2.5 of [5] the maps R and F agree up to homotopy on the image of the trace map from K-theory.

For, in their notation, there is a homotopy $D_p\circ\Delta_{p^n}\circ i=D_p\circ\Delta_p\circ\Delta_{p^{n-1}}\circ i\simeq\Delta_{p^{n-1}}\circ i$ and $\Phi_p=\Delta_p^{-1}$, so $\Phi_p\circ\Delta_{p^n}\circ i\simeq D_p\circ\Delta_{p^n}\circ i$. Here $\Delta_{p^n}\circ i$ induces the lifted trace map $\mathrm{tr}_{p^n}:K(A)\to T(A)^{C_{p^n}}$, D_p induces F, and Φ_p induces R. So in the current notation $R\circ\mathrm{tr}_{p^n}\simeq F\circ\mathrm{tr}_{p^n}$.

Thus, in the case $n=1$ of the diagram above, the trace map $\mathrm{tr}:K(\hat{\mathbb{Z}}_2)\to T(\mathbb{Z})_2$ factors as follows:

$$\begin{tikzcd} & K(\hat{\mathbb{Z}}_2)_2 \arrow[d, "\mathrm{tr}_2"'] \arrow[dr, "\mathrm{tr}"] & \\ (T(\mathbb{Z})_{hC_2})_2 \arrow[r, "N"] \arrow[d, equal] & T(\mathbb{Z})_2^{C_2} \arrow[r, "R"] \arrow[d, "\Gamma_1"] & T(\mathbb{Z})_2 \arrow[d, "\hat{\Gamma}_1"] \\ (T(\mathbb{Z})_{hC_2})_2 \arrow[r, "N^h"] & T(\mathbb{Z})_2^{hC_2} \arrow[r, "R^h"] & \hat{\mathbb{H}}(C_2, T(\mathbb{Z}))_2 \end{tikzcd} \tag{2.2}$$

We take these two diagrams as the definition of the norm and homotopy norm maps N and N^h, the homotopy restriction map R^h, and the comparison maps Γ_n and $\hat{\Gamma}_n$. We will use $\hat{\Gamma}_1:T(\mathbb{Z})\to\hat{\mathbb{H}}(C_2,T(\mathbb{Z}))$ to prove Theorem 0.2, by means of the following lemma.

Lemma 2.3. *If $\hat{\Gamma}_1$ is a connective two-adic equivalence, then so is Γ_1. Hence Theorem 0.2 follows if $\pi_*(\hat{\Gamma}_1;\mathbb{Z}/2)$ is an isomorphism for all $*\geq 0$.*

Proof. The map of fiber sequences above determines a homotopy equivalence from the homotopy fiber of Γ_1 to the homotopy fiber of $\hat{\Gamma}_1$, which proves the lemma. □

There is a spectral sequence $\hat{E}^*(G;\Lambda)$ for every closed subgroup $G \subseteq S^1$ and $\Lambda = \hat{\mathbb{Z}}_2$ or $\Lambda = \mathbb{Z}/2^v$, with E^2-term

$$(2.4) \qquad \hat{E}^2_{s,*} = \hat{H}^{-s}(G; T_*(\mathbb{Z};\Lambda)) \Rightarrow \pi_{s+*}(\hat{\mathbb{H}}(G, T(\mathbb{Z})); \Lambda).$$

Here $\hat{H}^*(G;M)$ denotes Tate cohomology [9] of G with coefficients in a G-module M. For a finite group G, $\hat{H}^k(G;M) = H^k(G;M)$ when $k \geq 1$, while $\hat{H}^{-k}(G;M) = H_{k-1}(G;M)$ when $k \geq 2$.

We write $\hat{E}^*(G) = \hat{E}^*(G;\hat{\mathbb{Z}}_2)$, to match our previous notation $E^*(G)$. As before the group action on $T_*(\mathbb{Z};\Lambda)$ is trivial, because the action extends through the path connected group S^1. We note that (2.4) is an upper half plane spectral sequence. Each of $\hat{E}^*(G)$ and $\hat{E}^*(G;\mathbb{Z}/2^v)$ with $v \geq 2$ is an algebra spectral sequence, when the E^2-term is given the product induced from the product on $T_*(\mathbb{Z};\Lambda)$ and the cup product in Tate cohomology. $\hat{E}^*(C_2;\mathbb{Z}/2)$ is not an algebra spectral sequence with this product on the E^2-term; see Remark 2.8. But we will see later that there is another "formal" algebra structure on this E^2-term, which does make $\hat{E}^*(C_2;\mathbb{Z}/2)$ into an algebra spectral sequence. See Theorem 4.1.

Recall that $\hat{H}^k(S^1;M) \cong M$ for $k \in \mathbb{Z}$ even, and 0 for k odd. Similarly $\hat{H}^k(C_{2^n};M) \cong M/2^n$ for k even and ${}_{2^n}M$ for k odd. Let $t \in \hat{H}^2(S^1;\mathbb{Z})$ be the generator compatible with our previous choice of $t \in H^2(S^1;\mathbb{Z})$. Now t is invertible in $\hat{H}^*(S^1;\mathbb{Z})$, with $t^{-1} \in \hat{H}^{-2}(S^1;\mathbb{Z})$. The class t maps to similar generators for all our $\hat{H}^2(G;\Lambda)$. We likewise extend the notations $u_n \in \hat{H}^1(C_{2^n};\mathbb{Z}/2)$ and $u'_n \in \hat{H}^1(C_{2^n};\mathbb{Z}/4)$.

The homotopy restriction map $R^h : T(\mathbb{Z})^{hG} \to \hat{\mathbb{H}}(G, T(\mathbb{Z}))$ is compatible with the map of spectral sequences from $E^*(G;\Lambda)$ to $\hat{E}^*(G;\Lambda)$ induced by the natural map $H^{-s}(G;M) \to \hat{H}^{-s}(G;M)$, which is the identity for $-s < 0$, the obvious surjection for $s = 0$, and the zero map for $-s > 0$. With $\mathbb{Z}/2$-coefficients, the E^2-terms take the following form:

$$\hat{E}^2_{s,*}(C_{2^n};\mathbb{Z}/2) \cong \mathbb{Z}/2[t, t^{-1}, u_n, e_3, e_4]/(u_n^2 = 0, e_3^2 = 0)$$

for $n \geq 2$, replacing $u_1^2 = 0$ with $u_1^2 = t$ when $n = 1$. On the level of E^2-terms, the homotopy restriction map R^h simply inverts t.

It may be helpful to make the mod four product pairing completely explicit.

Lemma 2.5. *In $\hat{E}^*(C_{2^n};\mathbb{Z}/4)$ we have $u'_1 \cdot f_3 = 0$ and $u'_1 \cdot f_4 = 0$ for $n = 1$, while $u'_n \cdot f_3 = u_n f_3$ and $u'_n \cdot f_4 = u_n f_4$ for $n \geq 2$. Here $u_n f_3$ generates $\hat{E}^2_{-1,3}(C_{2^n};\mathbb{Z}/4)$, and similarly for $u_n f_4$.*

Proof. u'_1 generates the order two torsion in $\mathbb{Z}/4$, i.e., the class of 2, and annihilates the order two classes f_3 and f_4. For $n \geq 2$, u'_n generates the order 2^n torsion in $\mathbb{Z}/4$, which is represented by the class $1 \in \mathbb{Z}/4$, and takes f_3 and f_4 to the generators of the order 2^n torsion in the $\mathbb{Z}/2$-groups they generate, i.e., to $u_n f_3$ and $u_n f_4$. □

$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	e_4^2	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$
$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	e_3e_4	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$
$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	e_4	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$
$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	e_3	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$
t^3	t^2u_1	t^2	tu_1	t	u_1	1	$t^{-1}u_1$	t^{-1}	$t^{-2}u_1$	t^{-2}	$t^{-3}u_1$	t^{-3}

Fig. 2.6. $\hat{E}^*(C_2;\mathbb{Z}/2) \Rightarrow \pi_*(\hat{\mathbb{H}}(C_2, T(\mathbb{Z}));\mathbb{Z}/2)$.

The E^2-term of $\hat{E}^*(C_2;\mathbb{Z}/2)$ appears as depicted in Fig. 2.6. Every nonzero group is $\mathbb{Z}/2$, with the listed generator.

Proposition 2.7. *In the spectral sequence $E^*(C_2;\mathbb{Z}/2)$ the classes $te_3 \in E^2_{-2,3}$ and $te_4 \in E^2_{-2,4}$ represent the images of the classes $i_1(\eta)$ and $\tilde{\eta}_2$ from $\pi_*(Q(S^0);\mathbb{Z}/2)$, respectively.*

In the spectral sequence $\hat{E}^*(C_2;\mathbb{Z}/2)$:

(1) $1 \in \hat{E}^2_{0,0}$ and $e_3 \in \hat{E}^2_{0,3}$ are hit by $i_1(1)$ and $i_1(\lambda)$ from $K_*(\mathbb{Z};\mathbb{Z}/2)$, and are thus infinite cycles.

(2) There is a nonzero differential $d^4(t^{-1}) = te_3$.

(3) There is a nonzero differential $d^5(t^{-2}u_1) = te_4$. In particular $d^4(t^{-2}u_1) = 0$.

The differential $d^4(t^{-1}) = te_3$ lifts over the coefficient reduction map i_1 to the spectral sequence $\hat{E}^*(C_2)$, and over the group restriction $C_2 \subset S^1$ to $\hat{E}^*(S^1)$. The latter two are algebra spectral sequences, and thus $d^4(t) = t^3g_3$ in both cases.

Proof. The initial claim and statement (1) is clear, by Theorem 1.5 and naturality with respect to the coefficient reduction map $i_1 : S^0 \to S^0/2$ and group restriction over $C_2 \subset S^1$.

For claim (2), consider where $i_1(\eta) \in \pi_1(Q(S^0);\mathbb{Z}/2) \to K_1(\hat{\mathbb{Z}}_2;\mathbb{Z}/2)$ maps in diagram (2.2). The class $i_1(\eta)$ maps to zero in $T_1(\mathbb{Z};\mathbb{Z}/2)$, so its image under $\hat{\Gamma}_1$ in $\hat{E}^*(C_2;\mathbb{Z}/2)$ is an infinite cycle that does not survive to E^∞, i.e., it must be a boundary. The factorization of tr through $T(\mathbb{Z})_2^{C_2}$ maps $i_1(\eta)$ to te_3, and R^h takes $te_3 \in E^2_{-2,3}(C_2;\mathbb{Z}/2)$ to the matching class $te_3 \in \hat{E}^2_{-2,3}(C_2;\mathbb{Z}/2)$. Thus te_3 is a boundary in $\hat{E}^*(C_2;\mathbb{Z}/2)$, and by bidegree considerations the only possible differential hitting this class is $d^4(t^{-1}) = te_3$.

Similar considerations for $\tilde{\eta}_2 \in \pi_2(Q(S^0);\mathbb{Z}/2)$ mapping to zero in $T_2(\mathbb{Z};\mathbb{Z}/2)$ and landing at te_4 in $E^*(C_2;\mathbb{Z}/2)$ show that te_4 is a boundary in $\hat{E}^*(C_2;\mathbb{Z}/2)$. The only possibilities for differentials are $d^2(e_3) = te_4$ and $d^5(t^{-2}u_1) = te_4$. The first is excluded since e_3 is an infinite cycle. Thus $t^{-2}u_1$ survives to E^5, but no longer.

The concluding claims are clear by naturality, combined with the following calculation in $\hat{E}^*(G)$:

$$0 = d^4(t \cdot t^{-1}) = d^4(t) \cdot t^{-1} + t \cdot d^4(t^{-1}) = d^4(t) \cdot t^{-1} + t^2 g_3$$

which gives $d^4(t) = t^3 g_3$. □

Remark 2.8. The final claim completes the proof of Theorem 1.9(2), which was postponed. It also makes it clear that $E^*(C_2; \mathbb{Z}/2)$ and $\hat{E}^*(C_2; \mathbb{Z}/2)$ are not algebra spectral sequences with the E^2-terms given, because u_1 survives to E^4 in both cases, but $d^4(t) = d^4(u_1^2) \neq 2u_1 \cdot d^4(u_1) = 0$.

In terms of spectra, we may express this by noting that $T(\mathbb{Z}) \wedge S^0/2$ admits a product map making it a ring spectrum up to homotopy, but this map cannot be chosen to be C_2-equivariant. This may be reformulated as stating that $1 \wedge \eta : T(\mathbb{Z}) \wedge S^1 \to T(\mathbb{Z})$ is inessential, but not C_2-equivariantly inessential.

Remark 2.9. Each map $TC(\mathbb{Z})_2 \xrightarrow{\pi} TF(\mathbb{Z})_2 \xrightarrow{F_n} T(\mathbb{Z})_2^{C_{2^n}}$ is a ring spectrum map, and $F : T(\mathbb{Z})_2^{C_{2^n}} \to T(\mathbb{Z})_2^{C_{2^{n-1}}}$ is a $TC(\mathbb{Z})_2$-module map. Since $F \circ F_n \circ \pi \simeq R \circ F_n \circ \pi$ the same goes for $R : T(\mathbb{Z})_2^{C_{2^n}} \to T(\mathbb{Z})_2^{C_{2^{n-1}}}$, and so the norm-restriction fiber sequence in (2.2) consists of $TC(\mathbb{Z})_2$-module spectra and maps. If trc is multiplicative then these are also $K(\hat{\mathbb{Z}}_2)$-module spectra and maps.

3. A double ladder of spectral sequences

In the next section we will determine the evolution of the spectral sequence

$$\hat{E}^2_{*,*}(C_2; \mathbb{Z}/2) \Rightarrow \pi_*(\hat{\mathbb{H}}(C_2, T(\mathbb{Z})); \mathbb{Z}/2),$$

by relating it to the spectral sequences $\hat{E}^*(C_{2^n}; \mathbb{Z}/2^v)$ for all $n \geq 1$ and $v = 1, 2$. Then we will conclude that $\hat{\Gamma}_1 : T(\mathbb{Z})_2 \to \hat{\mathbb{H}}(C_2, T(\mathbb{Z}))$ induces isomorphisms on mod two homotopy in non-negative degrees by a comparison with the corresponding map $\hat{\Gamma}_1 : T(\mathbb{F}_2) \to \hat{\mathbb{H}}(C_2, T(\mathbb{F}_2))$ for the prime field $\mathbb{F}_2$, which is proved to be a connective two-adic equivalence in [11].

Here we begin this program by reviewing a system of natural maps linking these spectral sequences together. We next derive vanishing results for the differentials in these spectral sequences.

Recall from Section 2 of [11] that the Frobenius map $F = F_n : T(\mathbb{Z})^{C_{2^{n+1}}} \to T(\mathbb{Z})^{C_{2^n}}$ is the inclusion forgetting part of the group action, and that there is a Verschiebung map $V = V_n : T(\mathbb{Z})^{C_{2^n}} \to T(\mathbb{Z})^{C_{2^{n+1}}}$ defined up to homotopy as a group transfer over the inclusion $C_{2^n} \subset C_{2^{n+1}}$. The key point is that the fixed point spectra T^{C_q} are identified with $\mathrm{Map}((S^1/C_q)_+, T)^{S^1}$ when $C_q \subset S^1$ and $T = T(\mathbb{Z})$ is an S^1-spectrum. F_n is then induced by the S^1-map $(S^1/C_{2^n})_+ \to (S^1/C_{2^{n+1}})_+$ while V_n is induced by the (stably defined) S^1-equivariant transfer map $(S^1/C_{2^{n-1}})_+ \to (S^1/C_{2^n})_+$.

We might also mention that F is multiplicative, F and V satisfy Frobenius reciprocity, and that $FV = VF = 2$.

There are similar Frobenius and Verschiebung maps linking the homotopy fixed point spectra, given by acting upon $T = \mathrm{Map}(ES^1_+, T(\mathbb{Z}))$. Also there are corresponding maps of Tate constructions

$$F = F_n : \hat{\mathbb{H}}(C_{2^{n+1}}, T(\mathbb{Z})) \to \hat{\mathbb{H}}(C_{2^n}, T(\mathbb{Z})),$$
$$V = V_n : \hat{\mathbb{H}}(C_{2^n}, T(\mathbb{Z})) \to \hat{\mathbb{H}}(C_{2^{n+1}}, T(\mathbb{Z})),$$

given by acting upon $T = \tilde{E}S^1 \wedge \mathrm{Map}(ES^1_+, T(\mathbb{Z}))$. All these Frobenius maps (resp. Verschiebung maps) are compatible under Γ_n and R^h, since Γ_n is induced by the natural map $c : T(\mathbb{Z}) \to \mathrm{Map}(ES^1_+, T(\mathbb{Z}))$, and R^h in turn is induced by the map $S^0 \to \tilde{E}S^1$.

F_n and V_n induce maps of spectral sequences

$$\hat{E}^*(C_{2^{n+1}}; \Lambda) \underset{V_n}{\overset{F_n}{\rightleftarrows}} \hat{E}^*(C_{2^n}; \Lambda). \tag{3.1}$$

On E^2-terms F_n is induced by the group restriction

$$\hat{H}^*(C_{2^{n+1}}; M) \to \hat{H}^*(C_{2^n}; M)$$

on Tate cohomology, compatible with the group restriction map in ordinary group cohomology. Similarly V_n is induced by the group transfer

$$\hat{H}^*(C_{2^n}; M) \to \hat{H}^*(C_{2^{n+1}}; M)$$

on Tate cohomology, compatible with the group inclusion map in ordinary group homology. The following calculations are standard.

Lemma 3.2.

$$\hat{H}^*(C_{2^n}; \mathbb{Z}/2) \cong \mathbb{Z}/2[t, t^{-1}, u_n]/(u_n^2 = 0),$$
$$\hat{H}^*(C_{2^n}; \mathbb{Z}/4) \cong \mathbb{Z}/4[t, t^{-1}, u'_n]/((u'_n)^2 = 0)$$

for $n \geq 2$, while

$$\hat{H}^*(C_2; \mathbb{Z}/2) \cong \mathbb{Z}/2[t, t^{-1}, u_1]/(u_1^2 = t),$$
$$\hat{H}^*(C_2; \mathbb{Z}/4) \cong \mathbb{Z}/2[t, t^{-1}, u'_1]/((u'_1)^2 = 0).$$

When $\Lambda = \mathbb{Z}/2$, the maps F_n and V_n are given on Tate cohomology by $F_n(u_{n+1}) = 0$, $F_n(t) = t$, $V_n(u_n) = u_{n+1}$, and $V_n(t) = 0$.

For $\Lambda = \mathbb{Z}/4$ and $n = 1$ the maps F_1 and V_1 are given by $F_1(u'_2) = u'_1$, $F_1(t) = t$, $V_1(u'_1) = 2u'_2$ and $V_1(t) = 2t$.

Finally, when $\Lambda = \mathbb{Z}/4$ and $n \geq 2$ the maps F_n and V_n are given by $F_n(u'_{n+1}) = 2u'_n$, $F_n(t) = t$, $V_n(u'_n) = u'_{n+1}$ and $V_n(t) = 2t$.

Hence the E^2-terms in (3.1) take the following form:

$$\hat{E}^2_{s,*}(C_{2^n};\mathbb{Z}/2)\cong\begin{cases}\mathbb{Z}/2[e_4]\{t^k,t^ke_3\} & \text{for } -s=2k,\\ \mathbb{Z}/2[e_4]\{t^ku_n,t^ku_ne_3\} & \text{for } -s=2k+1\end{cases}$$

for all $n\geq 1$, and

$$\hat{E}^2_{s,*}(C_2;\mathbb{Z}/4)\cong\begin{cases}\mathbb{Z}/2[f_8]\{t^k,t^kf_3,t^kf_4,t^kf_7\} & \text{for } -s=2k,\\ \mathbb{Z}/2[f_8]\{t^ku_1',t^ku_1f_3,t^ku_1f_4,t^ku_1'f_7\} & \text{for } -s=2k+1,\end{cases}$$

while

$$\hat{E}^2_{s,*}(C_{2^n};\mathbb{Z}/4)$$
$$\cong\begin{cases}\mathbb{Z}/4[f_8]\{t^k,t^kf_7\}\oplus\mathbb{Z}/2[f_8]\{t^kf_3,t^kf_4\} & \text{for } -s=2k,\\ \mathbb{Z}/4[f_8]\{t^ku_n',t^ku_n'f_7\}\oplus\mathbb{Z}/2[f_8]\{t^ku_nf_3,t^ku_nf_4\} & \text{for } -s=2k+1\end{cases}$$

for $n\geq 2$. The group comparison maps on E^2-terms are given as follows.

Lemma 3.3. *Suppose $\Lambda=\mathbb{Z}/2$. Then $F_n(t^ke_4^\ell x)=t^ke_4^\ell F_n(x)$ and $V_n(t^ke_4^\ell x)=t^ke_4^\ell V_n(x)$ for all integers k and $\ell\geq 0$. The Frobenius map satisfies $F_n(1)=1$, $F_n(e_3)=e_3$, $F_n(u_{n+1})=0$ and $F_n(u_{n+1}e_3)=0$, while the Verschiebung map is determined by $V_n(1)=0, V_n(e_3)=0, V_n(u_n)=u_{n+1}$ and $V_n(u_ne_3)=u_{n+1}e_3$.*

Suppose $\Lambda=\mathbb{Z}/4$ and $n=1$. Then $F_1(t^kf_8^\ell x)=t^kf_8^\ell F_1(x)$ and $V_1(t^kf_8^\ell x)=t^kf_8^\ell V_1(x)$ for all integers k and $\ell\geq 0$. Now $F_1(1)=1$, $F_1(f_3)=f_3$, $F_1(f_4)=f_4$ and $F_1(f_7)=f_7$, while $F_1(u_2')=u_1'$, $F_1(u_2f_3)=0$, $F_1(u_2f_4)=0$ and $F_1(u_2'f_7)=u_1'f_7$. Further $V_1(1)=2$, $V_1(f_3)=0$, $V_1(f_4)=0$ and $V_1(f_7)=2f_7$, while $V_1(u_1')=2u_2'$, $V_1(u_1f_3)=u_2f_3$, $V_1(u_1f_4)=u_2f_4$ and $V_1(u_1'f_7)=2u_2'f_7$.

Suppose $\Lambda=\mathbb{Z}/4$ and $n\geq 2$. Still $F_n(t^kf_8^\ell x)=t^kf_8^\ell F_n(x)$, and $V_n(t^kf_8^\ell x)=t^kf_8^\ell V_n(x)$. Here $F_n(1)=1$, $F_n(f_3)=f_3$, $F_n(f_4)=f_4$ and $F_n(f_7)=f_7$, while $F_n(u_{n+1}')=2u_n'$, $F_n(u_{n+1}f_3)=0$, $F_n(u_{n+1}f_4)=0$ and $F_n(u_{n+1}'f_7)=2u_n'f_7$. Finally $V_n(1)=2$, $V_n(f_3)=0$, $V_n(f_4)=0$ and $V_n(f_7)=2f_7$, while $V_n(u_n')=u_{n+1}'$, $V_n(u_nf_3)=u_{n+1}f_3$, $V_n(u_nf_4)=u_{n+1}f_4$ and $V_n(u_n'f_7)=u_{n+1}'f_7$.

Corollary 3.4. *In the diagram* (3.1), *F_n induces an isomorphism between the even columns of the E^2-terms, when $n\geq 1$ and $\Lambda=\mathbb{Z}/2$ or $\mathbb{Z}/4$.*

Similarly, V_n induces an isomorphism between the odd columns of the E^2-terms, when $\Lambda=\mathbb{Z}/2$ and $n\geq 1$, or when $\Lambda=\mathbb{Z}/4$ and $n\geq 2$.

The coefficient reduction and extension maps induced by ρ: $S^0/4\to S^0/2$ and $\varepsilon: S^0/2\to S^0/4$ induce maps of spectral sequences

$$\hat{E}^*(C_{2^n};\mathbb{Z}/4)\underset{\varepsilon}{\overset{\rho}{\rightleftarrows}}\hat{E}^*(C_{2^n};\mathbb{Z}/2) \tag{3.5}$$

for all $n \geq 1$. The actions of ρ and ε on $T_*(\mathbb{Z};\Lambda)$ were described in Theorem 1.6(3) and (5). Upon applying Tate cohomology we get the following coefficient comparison maps on the level of E^2-terms.

Lemma 3.6. *First suppose $n=1$. Then $\rho(t^k f_8^\ell x) = t^k e_4^{2\ell}\rho(x)$, and $\varepsilon(t^k e_4^{2\ell} x) = t^k f_8^\ell \varepsilon(x)$ for all integers k and $\ell \geq 0$. The coefficient reduction map satisfies $\rho(1)=1$, $\rho(f_3)= e_3, \rho(f_4)=0$ and $\rho(f_7)=e_3e_4$, while $\rho(u_1')=0$, $\rho(u_1 f_3)=u_1e_3$, $\rho(u_1 f_4)=0$ and $\rho(u_1' f_7) =0$. The coefficient extension map satisfies $\varepsilon(1)=0$, $\varepsilon(e_3)=0$, $\varepsilon(e_4)=f_4$ and $\varepsilon(e_3e_4) =0$, while $\varepsilon(u_1)=u_1'$, $\varepsilon(u_1e_3)=0$, $\varepsilon(u_1e_4)=u_1f_4$ and $\varepsilon(u_1e_3e_4)=u_1'f_7$.*

Now suppose $n \geq 2$. Still $\rho(t^k f_8^\ell x)=t^k e_4^{2\ell}\rho(x)$, and $\varepsilon(t^k e_4^{2\ell}x)=t^k f_8^\ell \varepsilon(x)$ for all integers k and $\ell \geq 0$. The coefficient reduction map satisfies $\rho(1)=1$, $\rho(f_3)=e_3$, $\rho(f_4)=0$ and $\rho(f_7)=e_3e_4$, while $\rho(u_n')=u_n$, $\rho(u_nf_3)=u_ne_3$, $\rho(u_nf_4)=0$ and $\rho(u_n'f_7)=u_ne_3e_4$. The coefficient extension map satisfies $\varepsilon(1)=2$, $\varepsilon(e_3)=0$, $\varepsilon(e_4)=f_4$ and $\varepsilon(e_3e_4)= 2f_7$, while $\varepsilon(u_n)=2u_n'$, $\varepsilon(u_ne_3)=0$, $\varepsilon(u_ne_4)=u_nf_4$ and $\varepsilon(u_ne_3e_4)=2u_n'f_7$.

Altogether we obtain the following double ladder of spectral sequences:

$$
\begin{tikzcd}
\hat{E}^*(C_2;\mathbb{Z}/2) \arrow[r, shift left, "V_1"] \arrow[d, shift right, "\rho"'] & \hat{E}^*(C_4;\mathbb{Z}/2) \arrow[l, shift left, "F_1"] \arrow[r, shift left, "V_2"] \arrow[d, shift right, "\rho"'] & \hat{E}^*(C_8;\mathbb{Z}/2) \arrow[l, shift left, "F_2"] \arrow[r, shift left, "V_3"] \arrow[d, shift right, "\rho"'] & \cdots \arrow[l, shift left, "F_3"] \\
\hat{E}^*(C_2;\mathbb{Z}/4) \arrow[r, shift left, "V_1"] \arrow[u, shift right, "\varepsilon"'] & \hat{E}^*(C_4;\mathbb{Z}/4) \arrow[l, shift left, "F_1"] \arrow[r, shift left, "V_2"] \arrow[u, shift right, "\varepsilon"'] & \hat{E}^*(C_8;\mathbb{Z}/4) \arrow[l, shift left, "F_2"] \arrow[r, shift left, "V_3"] \arrow[u, shift right, "\varepsilon"'] & \cdots \arrow[l, shift left, "F_3"]
\end{tikzcd}
$$

There is a similar double ladder of spectral sequences $E^*(C_{2^n};\Lambda)$, essentially obtained by truncating to the upper left quadrant, and the maps $R^h : E^*(C_{2^n};\Lambda) \to \hat{E}^*(C_{2^n};\Lambda)$ induce a map of double ladders.

We can now deduce some systematic results.

Suppose $\Lambda=\mathbb{Z}/2$, fix an integer $n_0 \geq 1$, and consider the family of spectral sequences $\hat{E}^*(C_{2^n};\mathbb{Z}/2)$ with $n \geq n_0$. Let r_0 be the minimal odd r such that there is a nonzero d^r-differential in some of these spectral sequences. (If there are none, let $r_0=\infty$.) So the only nonzero d^r in $\hat{E}^*(C_{2^n};\mathbb{Z}/2)$ with $n \geq n_0$ and $r<r_0$ appear for r even. In particular there is no interaction between the odd and even columns. Thus the E^2-isomorphisms of Corollary 3.4 propagate up to and including the E^{r_0}-terms. Hence all $\hat{E}^{r_0}(C_{2^n};\mathbb{Z}/2)$ for $n \geq n_0$ are abstractly isomorphic, with the isomorphisms induced by the F_n on even columns, and by the V_n on odd columns.

Similar considerations apply with $\Lambda=\mathbb{Z}/4$, if we assume $n_0 \geq 2$. Again there will be a maximal odd r_0 such that $d^r=0$ for r odd, $r<r_0$, in all $\hat{E}^*(C_{2^n};\mathbb{Z}/4)$ with $n \geq n_0$, and F_n and V_n induce isomorphisms between the even and odd columns, respectively, of all these spectral sequences up through the E^{r_0}-term.

Definition 3.7. Let $r_0 = r_0(n_0, v)$ be the length of the first nontrivial odd differential among the spectral sequences $\hat{E}^*(C_{2^n}; \mathbb{Z}/2^v)$ with $n \geq n_0$. When $v = 1$ (so $\Lambda = \mathbb{Z}/2$) we also write $r_0(n_0) = r_0(n_0, 1)$.

Proposition 3.8. *The shortest nonzero odd differential among the $\hat{E}^*(C_{2^n}; \mathbb{Z}/2)$ with $n \geq n_0$, if any, appears as a d^{r_0}-differential originating on $\hat{E}^{r_0}_{s,*}(C_{2^{n_0}}; \mathbb{Z}/2)$ with s odd. Then $d^{r_0} = 0$ on $\hat{E}^{r_0}_{s,*}(C_{2^{n_0}}; \mathbb{Z}/2)$ with s even, and $d^{r_0} = 0$ on all $\hat{E}^{r_0}(C_{2^n}; \mathbb{Z}/2)$ with $n > n_0$.*

In particular $r_0(n_0)$ is strictly increasing in n_0 (while finite).

The shortest nonzero odd differential among the $\hat{E}^*(C_{2^n}; \mathbb{Z}/4)$ with $n \geq n_0 \geq 2$, if any, appears as a d^{r_0}-differential originating on $\hat{E}^{r_0}_{s,*}(C_{2^{n_0}}; \mathbb{Z}/4)$ with s odd. Then $d^{r_0} = 0$ on $\hat{E}^{r_0}_{s,*}(C_{2^n}; \mathbb{Z}/4)$ if s is even and $n \geq n_0$, or if s is odd and $n \geq n_0 + 2$. If s is odd and $n = n_0 + 1$, then $2 \cdot d^{r_0} = 0$ when $* \equiv 0, 7 \bmod 8$, while $d^{r_0} = 0$ if $* \equiv 3, 4 \bmod 8$.

Proof. Consider the following diagram, with r_0 odd and $n \geq n_0$.

$$\begin{tikzcd}
\hat{E}^{r_0}_{s,*}(C_{2^{n+1}}; \mathbb{Z}/2) \arrow[r, "d^{r_0}"] \arrow[d, shift left, "F_n"] & \hat{E}^{r_0}_{s-r_0,*}(C_{2^{n+1}}; \mathbb{Z}/2) \arrow[d, shift left, "F_n"] \\
\hat{E}^{r_0}_{s,*}(C_{2^n}; \mathbb{Z}/2) \arrow[r, "d^{r_0}"] \arrow[u, shift left, "V_n"] & \hat{E}^{r_0}_{s-r_0,*}(C_{2^n}; \mathbb{Z}/2). \arrow[u, shift left, "V_n"]
\end{tikzcd}$$

If s is odd, $s - r_0$ is even, and V_n is an isomorphism on the left while it is zero on the right. Hence the top d^{r_0} is zero. So $d^{r_0} = 0$ on $\hat{E}^{r_0}_{s,*}(C_{2^n}; \mathbb{Z}/2)$ for s odd and $n > n_0$.

On the other hand, if s is even, $s - r_0$ is odd, and F_n is an isomorphism on the left while it is zero on the right. Thus the bottom d^{r_0} is zero. So $d^{r_0} = 0$ on $\hat{E}^{r_0}_{s,*}(C_{2^n}; \mathbb{Z}/2)$ for s even and $n \geq n_0$.

Hence the only possible nonzero d^{r_0}-differentials are as claimed, originating in $\hat{E}^{r_0}_{s,*}(C_{2^{n_0}}; \mathbb{Z}/2)$ for s odd.

The proof in the case of $\mathbb{Z}/4$-coefficients is similar, except that instead of being zero in even columns V_n is zero in fiber degrees $* \equiv 3, 4 \bmod 8$ and multiplies by two when $* \equiv 0, 7 \bmod 8$, and likewise for F_n in odd columns. The composites $V_{n+1} \circ V_n$ and $F_n \circ F_{n+1}$ do induce zero maps in these columns. □

Remark 3.9. For a fixed n, the spectral sequence $\hat{E}^*(C_{2^n}; \mathbb{Z}/2)$ is abstractly isomorphic (through the Frobenius and Verschiebung maps) to each of the preceding spectral sequences $\hat{E}^*(C_{2^i}; \mathbb{Z}/2)$ for $1 \leq i \leq n$ up through the E^r-term, with $r = r_0(i)$. This range increases up to $r_0(n)$ as i grows to n. Hence we have for each $i \geq 1$ a well-defined abstract segment of a spectral sequence, given by the E^r-terms and differentials of $\hat{E}^*(C_{2^i}; \mathbb{Z}/2)$ for $r_0(i-1) < r \leq r_0(i)$. Each of these segments has only even nontrivial differentials.

The spectral sequence $\hat{E}^*(C_{2^n};\mathbb{Z}/2)$ evolves through the first n of these segments, for $1 \le i \le n$. Then its first odd differential d^{r_0} with $r_0 = r_0(n)$ appears, and distinguishes $\hat{E}^*(C_{2^n};\mathbb{Z}/2)$ from the subsequent spectral sequences. It will appear in the second part of this paper that with $\mathbb{Z}/2$-coefficients, $\hat{E}^*(C_{2^n};\mathbb{Z}/2)$ collapses at the E^{r_0+1}-term, immediately after the first odd differential. Thus $\hat{E}^*(C_{2^n};\mathbb{Z}/2)$ passes through n systematic stages of even differentials, followed by one terminating odd differential.

Similar considerations apply with $\mathbb{Z}/4$-coefficients, except that the spectral sequence $\hat{E}^*(C_2;\mathbb{Z}/4)$ is somewhat exceptional, and that these spectral sequences do not collapse immediately after their first odd differential.

Next we have some vanishing results. Consider the square of spectral sequences with $\Lambda = \mathbb{Z}/2^v$:

$$\begin{array}{ccc} \hat{E}^*(S^1) & \longrightarrow & \hat{E}^*(C_{2^n}) \\ \big\downarrow & & \big\downarrow \\ \hat{E}^*(S^1;\Lambda) & \longrightarrow & \hat{E}^*(C_{2^n};\Lambda) \end{array}$$

Proposition 3.10. (1) *The only nonzero differentials in $\hat{E}^*(S^1)$ come from the horizontal axis.*

(2) *Let $\Lambda = \mathbb{Z}/2^v$ with $v \ge 1$. The classes in (even, odd) bidegrees in $\hat{E}^*(S^1;\Lambda)$ are infinite cycles, and the classes in (odd, odd) bidegrees are never boundaries.*

(3) *Let $n \ge 1$. The classes in (even, odd) bidegrees in $\hat{E}^*(C_{2^n})$ are infinite cycles, and the classes in (odd, odd) bidegrees are never boundaries.*

(4) *$d^r = 0$ for $r \equiv 2, 3 \bmod 4$ in $\hat{E}^r_{s,*}(C_{2^n};\Lambda)$, with a possible exception in the cases when $r \equiv 2 \bmod 4$, s is odd and $* \equiv 3 \bmod 4$.*

Proof. (1) Above the horizontal axis $\hat{E}^*(S^1)$ is concentrated in (even, odd) bidegrees, so for bidegree reasons there cannot be any nonzero differentials originating from positive fiber degrees. Hence all classes above the axis are infinite cycles.

(2) $\hat{E}^*(S^1;\Lambda)$ is concentrated in the even columns, and on E^2-terms the coefficient reduction map $\hat{E}^*(S^1) \to \hat{E}^*(S^1;\Lambda)$ is surjective in odd fiber degrees. Let x be a class surviving to $\hat{E}^r_{s,*}(S^1;\Lambda)$ in an (even, odd) bidegree. Then x also survives in the truncated spectral sequence where all groups in filtration degrees greater than s are set to zero. And a lift of x to the E^2-term will be the image of a class y from $\hat{E}^2_{s,*}(S^1)$. Then y is an infinite cycle, and in the truncated spectral sequence there is no room for it to be a boundary, so y will survive to E^r and map to x. Hence x is also an infinite cycle.

A class in an (even, even) bidegree could only be the boundary of a differential from an (even, odd) bidegree, where we have just seen that all the classes are infinite cycles.

(3) The group restriction map $\hat{E}^*(S^1) \to \hat{E}^*(C_{2^n})$ is surjective in (even, odd) bidegrees on E^2-terms, so again by considering truncated spectral sequences it follows that each class surviving to $\hat{E}^r(C_{2^n})$ in these bidegrees is the image of an infinite cycle, and thus does not support a differential.

Then a class in an (odd, odd) bidegree can only be hit from an (even, odd) bidegree, or from the horizontal axis, and the first possibility has just been excluded. On the other hand, a differential from the horizontal axis would have to originate in an odd filtration degree, and $\hat{H}^{-s}(C_{2^n}; \hat{\mathbb{Z}}_2) = 0$ for s odd. So also this second kind of differential must be zero.

(4) Consider $d^r : \hat{E}^r_{s,*} \to \hat{E}^r_{s-r,*+r-1}$. From bidegree considerations it is clear that $d^r = 0$ when $r \equiv 3 \bmod 4$, or when $r \equiv 2 \bmod 4$ and $* \not\equiv 3 \bmod 4$. So let $x \in \hat{E}^r_{s,*}(C_{2^n}; \Lambda)$ and assume $r \equiv 2 \bmod 4$ and $* \equiv 3 \bmod 4$.

Supposing s is even, $\hat{E}^2(S^1) \to \hat{E}^2(C_{2^n}; \Lambda)$ is surjective in bidegree $(s, *)$, so in a truncated spectral sequence x lifts to an infinite cycle in $\hat{E}^*(S^1)$, which survives to the E^r-term. Thus $d^r(x) = 0$. □

Proposition 3.11. *$d^r = 0$ on $\hat{E}^*(C_{2^n}; \mathbb{Z}/2^v)$ for $r \equiv 2 \bmod 4$, $n \geq 1$ and $v = 1$ or 2, as long as d^r is shorter than the first nontrivial odd differential in this spectral sequence.*

Proof. Fix $r \equiv 2 \bmod 4$ with $r < r_0(n, v)$. By Proposition 3.10(4) it suffices to show that $d^r(x) = 0$ for $x \in \hat{E}^r_{s,*}(C_{2^n}; \mathbb{Z}/2^v)$ with s odd and $* \equiv 3 \bmod 4$. Fix attention on such a bidegree $(s, *)$. We shall first show that $d^r(x) = 0$ when n is sufficiently large. Thereafter we use that the Verschiebung maps V_n are isomorphisms in odd columns in the given range, except for V_1 with $\mathbb{Z}/4$-coefficients. Hence $d^r(x) = 0$ also for smaller n, when $v = 1$ and $n \geq 1$ or $v = 2$ and $n \geq 2$. We handle the exceptional case of $\hat{E}^*(C_2; \mathbb{Z}/4)$ separately at the end.

Let $\pi : \mathbb{Z} \to \mathbb{Z}/2^v$ be the ring surjection and consider the maps of spectral sequence terms

$$\hat{E}^2(C_{2^n}) \xrightarrow{E^2(\pi)} \hat{E}^2(C_{2^n}; \mathbb{Z}/2^v),$$
$$\hat{E}^r(C_{2^n}) \xrightarrow{E^r(\pi)} \hat{E}^r(C_{2^n}; \mathbb{Z}/2^v).$$

Let $M = T_*(\mathbb{Z})_2 \cong \mathbb{Z}/2^m$ in the given fiber degree. Then $E^2_{s,*}(\pi)$ is a map $\hat{H}^{-s}(C_{2^n}; M) \to \hat{H}^{-s}(C_{2^n}; M/2^v)$, which for s odd is the natural homomorphism ${}_{2^n}(M) \to {}_{2^n}(M/2^v)$. This map is surjective if (and only if) $m \leq \max\{n, v\}$. Hence $E^2_{s,*}(\pi)$ is surjective for n sufficiently large.

In the odd columns of $\hat{E}^*(C_{2^n})$ we have $\hat{E}^2(C_{2^n}) = \hat{E}^r(C_{2^n})$ and $d^r = 0$ for $r < r_0(n, v)$. For $\hat{E}^*(C_{2^n})$ is concentrated in odd fiber degrees above the horizontal axis, so all even differentials originating above the horizontal axis are trivial. Furthermore the classes on the horizontal axis have even filtration degree, so the even differentials supported on them land in even columns.

Thus any class $x \in \hat{E}^r_{s,*}(C_{2^n}; \mathbb{Z}/2^v)$ is represented by a class $y \in \hat{E}^2_{s,*}(C_{2^n}; \mathbb{Z}/2^v)$ surviving to E^r, which for n sufficiently large lifts to a class $z \in \hat{E}^2_{s,*}(C_{2^n})$ with

$E^2(\pi)(z)=y$. Then z survives to E^r and $d^r(z)=0$. Hence $E^r(\pi)(z)=x$ and $d^r(x)=0$ by naturality.

The exceptional case remains. We assert that $d^2=0$ on $\hat{E}^*(C_2;\mathbb{Z}/4)$, and shall soon see that $r_0(1,2)=5$, so this claim will complete the proof. The Verschiebung map V_1 induces an isomorphism in odd columns and fiber degrees $*\equiv 3,4 \bmod 8$, when we use mod four coefficients, so $d^2=0$ when $*\equiv 3 \bmod 8$. Finally, when $*\equiv 7 \bmod 8$ the coefficient extension $\varepsilon:\hat{E}^2(C_2;\mathbb{Z}/2)\to\hat{E}^2(C_2;\mathbb{Z}/4)$ induces an isomorphism in fiber degrees $*\equiv 0,7 \bmod 8$, and since $d^2=0$ in the case $\Lambda=\mathbb{Z}/2$ the same holds with $\Lambda=\mathbb{Z}/4$ in these bidegrees. □

Thus only d^r-differentials with $r\equiv 0 \bmod 4$ need to be considered before the first odd differential appears, in all the spectral sequences under consideration, and the first odd differential (if any) is always of length $r\equiv 1 \bmod 4$.

4. The C_2-spectral sequence

The E^2-term of $\hat{E}^*(C_2;\mathbb{Z}/2)$ can be given a *formal algebra structure* as follows:

$$\hat{E}^*(C_2;\mathbb{Z}/2)\cong\mathbb{Z}/2[t,t^{-1},u_1,e_3,e_4]/(u_1^2=0,e_3^2=0)$$

This is different from the algebra structure composed from the subalgebra structure on $T_*(\mathbb{Z};\mathbb{Z}/2)$ inherited from $H_*^{spec}(T(\mathbb{Z});\mathbb{Z}/2)$, and the product on Tate cohomology, where the relation $u_1^2=0$ would have been replaced by $u_1^2=t$.

Theorem 4.1. *The spectral sequence $\hat{E}^*(C_2;\mathbb{Z}/2)$ is an algebra spectral sequence, in the sense that the differentials are derivations, when the E^2-term is given the formal algebra structure above.*

The d^2- and d^3-differentials are zero. The d^4-differentials are determined by $d^4(t)=t^3e_3$ and $d^4(u_1)=0$, and that e_3 and te_4 are infinite cycles. The d^5-differentials are determined by $d^5(u_1)=t^3e_4$, and that t^2 and t^{-2} are infinite cycles.

The spectral sequence collapses at the E^6-term, so $E^6=E^\infty$, and it converges additively to $\pi_*(\hat{\mathbb{H}}(C_2,T(\mathbb{Z}));\mathbb{Z}/2)$. Thus

$$\hat{E}^\infty(C_2;\mathbb{Z}/2)\cong\mathbb{Z}/2[t^2,t^{-2},e_3]/(e_3^2=0)$$

and so

$$\pi_*(\hat{\mathbb{H}}(C_2,T(\mathbb{Z}));\mathbb{Z}/2)\cong\begin{cases}\mathbb{Z}/2 & \textit{if } *\equiv 0,3 \bmod 4,\\ 0 & \textit{otherwise.}\end{cases}$$

Proof. We consider the double ladder of spectral sequences $\hat{E}^*(C_{2^n};\Lambda)$ with $n\geq 1$ and $\Lambda=\mathbb{Z}/2$ or $\mathbb{Z}/4$. By Propositions 3.10(4) and 3.11 we have $d^2=0$ and $d^3=0$ in each case. Thus the nonzero odd differentials in these spectral sequences have length at least 5, and so the Frobenius and Verschiebung maps induce abstract isomorphisms of E^4-terms and d^4-differentials between the $\hat{E}^4(C_{2^n};\mathbb{Z}/2)$ for $n \geq 1$, and likewise for the $\hat{E}^4(C_{2^n};\mathbb{Z}/4)$ when $n \geq 2$.

First we determine the d^4-differentials originating in even columns. We will freely use the formulas for F_n, V_n, ρ and ε from Lemmas 3.3 and 3.6.

By Proposition 2.7(2) $d^4(t^{-1}) = te_3$ in $\hat{E}^4(C_2;\mathbb{Z}/2)$, so via the F_n, which are isomorphisms on even columns, $d^4(t^{-1}) = te_3$ in all $\hat{E}^4(C_{2^n};\mathbb{Z}/2)$ for $n \geq 1$. By ρ, which is surjective in fiber degree zero and an isomorphism in fiber degree three, $d^4(t^{-1}) = tf_3$ in $\hat{E}^4(C_{2^n};\mathbb{Z}/4)$, both for $n = 1$ and for $n \geq 2$.

By the algebra structure in the $\mathbb{Z}/4$-spectral sequences, $d^4(t^i) = t^{i+2}f_3$ for i odd, and 0 for i even, in $\hat{E}^*(C_{2^n};\mathbb{Z}/4)$ for all $n \geq 1$. Via ρ it follows that $d^4(t^i) = t^{i+2}e_3$ for i odd, and 0 for i even, in $\hat{E}^*(C_{2^n};\mathbb{Z}/2)$ for all $n \geq 1$. Thus d^4 from the even columns of the horizontal axis is completely determined.

Next $\tilde{\eta}_4$ maps to tf_4 in $E^*(S^1;\mathbb{Z}/4)$ by Theorem 1.9(1), so tf_4 is an infinite cycle there, and in $\hat{E}^*(S^1;\mathbb{Z}/4)$. Since it survives to $\hat{E}^4(C_2;\mathbb{Z}/4)$, it is also an infinite cycle there, by the group restriction over $C_2 \subset S^1$. Thus tf_4 is also an infinite cycle in all the intermediate $\hat{E}^*(C_{2^n};\mathbb{Z}/4)$ for $n \geq 1$.

By the algebra structure on mod four homotopy it follows that $d^4(t^i f_4) = t^{i+2} f_3 f_4$ for i even, and 0 for i odd, in $\hat{E}^*(C_{2^n};\mathbb{Z}/4)$. The product $f_3 f_4 = 2f_7$ is nonzero in the E^4-term when $n \geq 2$, but is zero when $n = 1$.

Via ε, which is an isomorphism in fiber degree four and injective in fiber degree seven when we assume $n \geq 2$, we get $d^4(t^i e_4) = t^{i+2} e_3 e_4$ for i even, and 0 for i odd, in $\hat{E}^*(C_{2^n};\mathbb{Z}/2)$. Finally the same formulas hold for $n = 1$ in $\hat{E}^*(C_2;\mathbb{Z}/2)$ by comparison over F_1. Thus d^4 from the even columns and fiber degree four is completely determined.

$\lambda \in K_3(\mathbb{Z})_2$ maps to the permanent cycle f_3 in each $E^*(C_{2^n};\mathbb{Z}/4)$, by group and coefficient reduction from $E^*(S^1)$, using Theorem 1.5(2). So f_3 is an infinite cycle in each $\hat{E}^*(C_{2^n};\mathbb{Z}/4)$, and similarly for e_3 in $\hat{E}^*(C_{2^n};\mathbb{Z}/2)$.

Similarly $\tilde{v}_4 \in K_4(\hat{\mathbb{Z}}_2;\mathbb{Z}/4)$ maps to the infinite cycle $t^2 f_8$ in $E^*(S^1;\mathbb{Z}/4)$, by Theorem 1.9(5), and so by group reduction $t^2 f_8$ is an infinite cycle in each $\hat{E}^*(C_{2^n};\mathbb{Z}/4)$.

Thus the classes f_3 and $t^2 f_8$ act upon each spectral sequence $\hat{E}^r(C_{2^n};\mathbb{Z}/4)$, as long as the respective class survives to the E^r-term. In particular $d^r(f_3 \cdot x) = f_3 \cdot d^r(x)$ and $d^r(t^2 f_8 \cdot x) = t^2 f_8 \cdot d^r(x)$.

We now know that f_3 is a permanent cycle for all n, so the action of f_3 propagates through to E^∞. For the only possible differential affecting f_3 would be $d^4(t^2)$, and we have just seen that by the algebra structure on mod four homotopy $d^4(t^2) = 0$ in this case.

We also know, from bidegree considerations, that $t^2 f_8$ survives at least to the E^5-term. So $t^2 f_8$ acts upon the E^r-terms of $\hat{E}^*(C_{2^n};\mathbb{Z}/4)$ for all $n \geq 1$ when $r \leq 5$.

(In fact $t^2 f_8$ survives to E^9 in $\hat{E}^*(C_2;\mathbb{Z}/4)$ and to E^∞ for $n \geq 2$, so the action of $t^2 f_8$ on the former spectral sequence lasts through to the E^9-term. Furthermore $\hat{E}^*(C_2;\mathbb{Z}/4)$ collapses immediately after that stage. We will not need these facts, and omit their proof.)

By the module action of the mod four spectral sequences upon the mod two spectral sequences, it follows that f_3 acts upon $\hat{E}^r(C_{2^n};\mathbb{Z}/2)$ as multiplication by e_3 in the formal algebra structure, for all $n\geq 1$ and $r\leq\infty$.

Similarly, t^2f_8 acts upon $\hat{E}^r(C_{2^n};\mathbb{Z}/2)$ as multiplication by $t^2e_4^2$ in the formal algebra structure, for all $n\geq 1$, when t^2f_8 survives to the E^r-term of $\hat{E}^*(C_{2^n};\mathbb{Z}/4)$. So this holds at least for $r\leq 5$.

Combined with our results in fiber degrees zero and four, this completely determines the d^4-differentials from the even columns in the spectral sequences under consideration. In summary, d^4 acts as a derivation, with $d^4(t^{-1})=te_3$ and $d^4(te_4)=0$ in $\hat{E}^*(C_{2^n};\mathbb{Z}/2)$, while $d^4(t^{-1})=tf_3$ and $d^4(tf_4)=0$ in $\hat{E}^*(C_{2^n};\mathbb{Z}/4)$.

Next we turn to the odd columns.

By Proposition 2.7(3) $d^5(t^{-2}u_1)=te_4$ in $\hat{E}^*(C_2;\mathbb{Z}/2)$, so the first nonzero odd differential among the $\hat{E}^*(C_{2^n};\mathbb{Z}/2)$ with $n\geq 1$ is a d^5-differential. In the notation of Section 3, $r_0(1)=5$. Hence all d^5-differentials from even columns of $\hat{E}^*(C_2;\mathbb{Z}/2)$ are zero, and all d^5-differentials in $\hat{E}^*(C_{2^n};\mathbb{Z}/2)$ with $n\geq 2$ are zero. This was the content of Proposition 3.8.

So $d^4(t^{-2}u_1)=0$ in $\hat{E}^*(C_2;\mathbb{Z}/2)$. By V_n we get $d^4(t^{-2}u_n)=0$ in $\hat{E}^*(C_{2^n};\mathbb{Z}/2)$ for all $n\geq 1$.

By ρ we get $d^4(t^{-2}u_n')=0$ in $\hat{E}^*(C_{2^n};\mathbb{Z}/4)$ for all n. The algebra structure and $d^4(t^2)=0$ implies $d^4(u_n')=0$ for all n, and so $d^4(t^iu_n')=d^4(t^i)u_n$ when $n\geq 2$, while $d^4(t^iu_1')=0$ in the case $n=1$. These products were discussed in Lemma 2.5.

Also tf_4 is an infinite cycle, as above. So we compute

$$d^4(t^iu_nf_4)=d^4(t^{i-1}u_n')tf_4=d^4(t^{i-1})u_ntf_4=d^4(t^if_4)\cdot u_n'$$

in $\hat{E}^*(C_{2^n};\mathbb{Z}/4)$ for $n\geq 2$.

By ρ we get $d^4(t^iu_n)=d^4(t^i)u_n$ in $\hat{E}^*(C_{2^n};\mathbb{Z}/2)$ for $n\geq 2$, and by ε we get $d^4(t^iu_ne_4)=d^4(t^ie_4)u_n$ in the same spectral sequence, when $n\geq 2$.

By V_1 the same two formulas also hold for $n=1$.

So by ε we get $d^4(t^iu_1f_4)=t^{i+2}u_1'f_7\neq 0$ when i is even, and 0 when i is odd, in $\hat{E}^*(C_2;\mathbb{Z}/4)$.

This determines d^4 on fiber degrees zero and four in all the spectral sequences being considered. The behavior in the remaining degrees is determined by the action of the surviving cycles f_3 and t^2f_8, as in the even column case. The odd columns and odd fiber degrees of $\hat{E}^*(C_2;\mathbb{Z}/4)$ might be thought to be an exception, but for bidegree reasons all d^4-differentials are zero there.

In conclusion, d^4 acts as a derivation on all of $\hat{E}^*(C_{2^n};\mathbb{Z}/2)$ for $n\geq 1$, with respect to the formal algebra structure, and is determined by $d^4(t^{-1})=te_3$ and $d^4(u_n)=0$. We note that the differentials mapping between odd columns are abstractly isomorphic to those mapping between the even columns, only shifted one degree to the left.

Of course d^4 acts as a derivation on all $\hat{E}^*(C_{2^n};\mathbb{Z}/4)$, and for $n\geq 2$ it is determined by $d^4(u_n')=0$ and the algebra structure. The action of d^4 on $\hat{E}^*(C_2;\mathbb{Z}/4)$ is also completely known.

In particular t^2 and t^{-2} survive to $\hat{E}^8(C_{2^n};\mathbb{Z}/4)$ for all $n \geq 1$, and so act invertibly on all the E^r-terms considered for $r < 8$. An easy inspection shows that the action of $t^2 f_8$ is periodic on the E^5-terms, i.e., the multiplication by $t^2 f_8$ is injective. For the E^2-terms are $t^2 f_8$-periodic, and the d^4-differentials preserve this symmetry.

Next we turn to the d^5-differentials of $\hat{E}^*(C_{2^n};\Lambda)$ for $\Lambda = \mathbb{Z}/2$ and $\mathbb{Z}/4$.

The invertible action of t^2, and the periodic action of $t^2 f_8$, shows that it is sufficient to determine the d^5-differentials originating in bidegrees $(s, *)$ with $1 \leq s \leq 4$ and $0 \leq * \leq 7$. Furthermore the action of f_3 determines the behavior of differentials originating in fiber degrees three and seven from those originating in fiber degrees zero or four, except in the case of $\hat{E}^*(C_2;\mathbb{Z}/4)$.

We already noted that $d^5(t^{-2}u_1) = te_4 \neq 0$ in $\hat{E}^*(C_2;\mathbb{Z}/2)$. So $r_0(1) = 5$ and $d^5 = 0$ throughout $\hat{E}^*(C_{2^n};\mathbb{Z}/2)$ for $n \geq 2$.

We now focus on $\hat{E}^*(C_2;\mathbb{Z}/4)$, in fiber degree zero. $d^5(t^{-2}u_1') = tf_4$ by ε. $d^5(t^{-2}) = 0$ since the target group supports a nonzero d^4-differential. t^{-1} supported a d^4-differential, and does not survive to the E^5-term. So only $t^{-1}u_1'$ remains, and we claim that $d^5(t^{-1}u_1') = 0$.

To see this, note that $\tilde{v}_4 \mapsto t^2 f_8$ is an infinite cycle, but maps to zero in $T_*(\mathbb{Z};\mathbb{Z}/4)$, so cannot survive to E^∞, i.e., is a boundary. A differential hitting it must come from $t^{-1}u_1 f_4$ or $t^{-3}u_1'$. Now

$$d^5(t^{-1}u_2 f_4) = d^5(t^{-2}u_2' \cdot tf_4) = tf_4 \cdot tf_4 = 0$$

in $\hat{E}^*(C_4;\mathbb{Z}/4)$, so by V_1 we get $d^5(t^{-1}u_1 f_4) = 0$ back in $\hat{E}^*(C_2;\mathbb{Z}/4)$. Thus $d^9(t^{-3}u_1') = t^2 f_8$. In particular $t^2 f_8$ survives to the E^9-term. Hence $d^5(t^{-3}u_1') = 0$, and acting by t^2 we get $d^5(t^{-1}u_1') = 0$, as claimed.

d^5 from fiber degree zero for $\hat{E}^*(C_2;\mathbb{Z}/2)$ is now completely determined by noting that $d^5(t^{-2}) = 0$, using ε.

Next we consider $\hat{E}^*(C_4;\mathbb{Z}/4)$, beginning in fiber degree zero. $d^5(2u_2') = 0$ and $d^5(t^{-2}) = 0$ since the target groups are zero. $d^5(2t^{-1}) = 0$ by comparison with F_2, for $2t^{-1}$ survives to $\hat{E}^5_{-2,0}(C_8;\mathbb{Z}/4)$. And using F_1 we get $d^5(t^{-2}u_2') = tf_4$.

In fiber degree four we have computed $d^5(t^{-1}u_2 f_4) = 0$. Next $t^{-1}f_4$ is a d^5-boundary, while $t^{-2}u_2 f_4$ and $t^{-2}f_4$ support d^4-differentials. Thus $d^5 = 0$ from fiber degree four.

We return to $\hat{E}^*(C_2;\mathbb{Z}/4)$, in fiber degree four. We have already noted that $d^5(t^{-1}u_1 f_4) = 0$. The class $t^{-1}f_4$ is a d^5-boundary, and $t^{-2}u_1 f_4$ supports a d^4-differential, so it remains to consider $d^5(t^{-2}f_4)$. It is 0 or $u_1' f_8$, but $d^5(u_1' f_8) = t^2 f_8 \cdot tf_4 \neq 0$, so since $d^5 \circ d^5 = 0$ we can only have $d^5(t^{-2}f_4) = 0$. Hence $d^5 = 0$ in fiber degree four in $\hat{E}^*(C_{2^n};\mathbb{Z}/4)$ for $n = 1$ and 2.

It remains to consider the d^5-differentials in $\hat{E}^*(C_2;\mathbb{Z}/2)$ from fiber degree four. $t^{-1}e_4$ is a d^5-boundary, while $t^{-2}u_1 e_4$ and $t^{-2}e_4$ support d^4-differentials. We claim $d^5(t^{-1}u_1 e_4) = t^2 e_4^2$. For by Theorem 1.9(5) $\widetilde{2v_2} \in K_4(\hat{\mathbb{Z}}_2;\mathbb{Z}/2)$ maps to $t^2 e_4^2$, while it maps to zero in $T_4(\mathbb{Z};\mathbb{Z}/2)$, whence $t^2 e_4^2$ must be a boundary. For bidegree reasons it must be hit from $te_3 e_4$, $t^{-1}u_1 e_4$, $t^{-1}e_3$ or $t^{-3}u_1$. The first is excluded since $d^2 = 0$. The latter two die after the E^4-term. The claimed d^5-differential is the only remaining possibility.

8	4	5	5′	4	4	5	5′	4	4	5	5′	4	4
7	5	4′	4′	5	5′	4′	4′	5	5′	4′	4′	5	5′
4	5	4	4	5	5′	4	4	5	5′	4	4	5	5′
3	4′	4′	∞	4′	4′	5	∞	4′	4′	5	∞	4′	4′
0	4	5	∞	4	4	5	∞	4	4	5	∞	4	4
	−6	−5	−4	−3	−2	−1	0	1	2	3	4	5	6

Fig. 4.2. $\hat{E}^*(C_2;\mathbb{Z}/4) \Rightarrow \pi_*(\hat{\mathbb{H}}(C_2,T(\mathbb{Z}));\mathbb{Z}/2)$.

This completes the determination of the d^5-differentials of $\hat{E}^*(C_{2^n};\Lambda)$ from fiber degrees zero and four, when $n=1$ or 2 and $\Lambda=\mathbb{Z}/2$ or $\mathbb{Z}/4$. The actions by t^2, f_3 and $t^2 f_8$ determine the remaining d^5-differentials in these cases, except in the odd columns and odd fiber degrees of $\hat{E}^*(C_2;\mathbb{Z}/4)$.

Here $d^5(t^{-2}u_1 f_3)=tf_7$ by ρ, $d^5(t^{-1}u_1' f_7)=0$ by ε, and $t^{-2}u_1' f_7$ is a d^4-boundary. At the moment $d^5(t^{-1}u_1 f_3)$ remains undetermined by these naturality arguments. (However, one can prove that $t^{-1}u_1 f_3$ is an infinite cycle.)

With this exception, the d^5-structure is now settled. For $d^5=0$ everywhere in $\hat{E}^*(C_{2^n};\mathbb{Z}/4)$ with $n\geq 3$, essentially by Proposition 3.8. □

It is now high time to give a picture of the spectral sequences. Instead of drawing in all the differentials, we have labeled the source and target of each nontrivial d^r-differential by r and r', respectively, while ∞ denotes a class surviving to E^∞. When $n=1$ the group in each bidegree is either $\mathbb{Z}/2$ or trivial.

In the proof of Theorem 4.1 we determined the d^4- and d^5-differentials in Fig. 4.2, as well as the d^4-differentials and most of the d^5-differentials in Fig. 4.3. The remaining d^5-, d^8- and d^9-differentials in Fig. 4.3 are as indicated, but will not be needed for the proof of Theorem 0.2, and we omit the proofs.

The d^4-differentials of $E^*(C_2;\mathbb{Z}/2)$ determine d^4-differentials of $E^*(S^1;\mathbb{Z}/2)$, and thus some additive extensions in $E^*(S^1)\Rightarrow \pi_* T(\mathbb{Z})_2^{hS^1}$.

Corollary 4.4. *Consider the additive extension*

$$\mathbb{Z}/2^a \cong E^\infty_{2i-4,4j+3}(S^1) \to B \to E^\infty_{2i,4j-1}(S^1) \cong \mathbb{Z}/2^c$$

of $E^\infty_{*,*}(S^1)$. *When* $i\equiv j \bmod 2$ *the extension is split, so* $B\cong \mathbb{Z}/2^a\oplus\mathbb{Z}/2^c$. *When* $i\not\equiv j \bmod 2$ *the extension is cyclic, so* $B\cong\mathbb{Z}/2^{a+c}$. *Here* $i\leq 0$ *and* $j\geq 1$, *while* $a=v_2(j+1)+1$ *and* $c=v_2(j)+1$.

Proof. Consider $E^*(S^1)$ truncated to the columns $2i-4\leq s\leq 2i$, and its mod two analog $E^*(S^1;\mathbb{Z}/2)$. There is a nontrivial d^4-differential landing in bidegree $(2i-4,4j+3)$ precisely if $i\not\equiv j \bmod 2$. The classes surviving in total degree $2i+4j-1$ form a

8	4	5	9′	8	4	5	8	9	4	5	9′	8	4
7	5′	4′	8′	9	5′	4′	∞	8′	5′	4′	8′	9	5′
4	5′	4	8	8	5′	4	8	8	5′	4	8	8	5′
3	4′	5	∞	∞	4′	5	∞	∞	4′	5	∞	∞	4′
0	4	5	8	9	4	5	∞	8	4	5	8	9	4
	−6	−5	−4	−3	−2	−1	0	1	2	3	4	5	6

Fig. 4.3. $\hat{E}^*(C_2;\mathbb{Z}/2) \Rightarrow \pi_*(\hat{\mathbb{H}}(C_2,T(\mathbb{Z}));\mathbb{Z}/4)$.

composition series for $B/2$, so B is cyclic precisely when only the class in bidegree $(2i,4j-1)$ survives. Otherwise, $B/2$ has order four, whence B has two generators. □

The d^8-differentials in $\hat{E}^*(C_2;\mathbb{Z}/4)$ (indicated in Fig. 4.3) which originate in even columns will lift to $E^*(S^1;\mathbb{Z}/4)$, and correspond to additive extensions across eight filtration degrees in $E^{\infty}_{*,*}(S^1)$. We leave it to the interested reader to make the resulting extensions explicit.

We next wish to complete the proof of Theorem 0.2, by verifying the condition of Lemma 2.3 that $\hat{\Gamma}_1 : T(\mathbb{Z}) \to \hat{\mathbb{H}}(C_2,T(\mathbb{Z}))$ induces an isomorphism on mod two homotopy groups in nonnegative degrees.

To achieve this, we make a comparison with the analogous constructions related to the K-theory of the finite field $\mathbb{F}_2$, which were studied in [11]. The ring map $\pi : \mathbb{Z} \to \mathbb{F}_2$ induces a map of ring spectra with S^1-action $\pi : T(\mathbb{Z}) \to T(\mathbb{F}_2)$, and thus of the associated fixed point, homotopy fixed point and Tate construction spectra. Let $E^*(G,\mathbb{F}_2;A)$ and $\hat{E}^*(G,\mathbb{F}_2;A)$ denote the corresponding spectral sequences, abutting to $\pi_*(T(\mathbb{F}_2)^{hG};A)$ and $\pi_*(\hat{\mathbb{H}}(G,T(\mathbb{F}_2));A)$, for $G \subseteq S^1$ and A as above.

Lemma 4.5. *$T_*(\mathbb{F}_2) \cong \mathbb{Z}/2[x_2]$ with $x_2 \in T_2(\mathbb{F}_2)$.*

$T_*(\mathbb{F}_2;\mathbb{Z}/2) \cong \mathbb{Z}/2[x_1,x_2]/(x_1^2=0)$ in the subalgebra structure inherited from $H_*^{spec}(T(\mathbb{F}_2);\mathbb{Z}/2)$.

$T_*(\mathbb{F}_2;\mathbb{Z}/4) \cong \mathbb{Z}/2[y_1,y_2]/(y_1^2=0)$ in the mod four homotopy algebra structure. Here $\rho(y_1)=0$, $\rho(y_2)=x_2$, $\varepsilon(x_1)=y_1$ and $\varepsilon(x_2)=0$.

Proof. This all follows from Section 4 of [11], except the mod four algebra structure. Now $y_2^n = i_2(x_2^n) \neq 0$, and $j_2(y_1 y_2^n) = y_2^n \neq 0$, so $y_1 y_2^n \neq 0$. It remains to check that $y_1^2 = 0$, but $\rho(y_1^2) = m(y_1 \wedge \rho(y_1)) = 0$ and ρ is an isomorphism on $T_2(\mathbb{F}_2;\mathbb{Z}/4)$. □

Proposition 4.6. (1) *The spectral sequence $\hat{E}^*(C_2,\mathbb{F}_2;\mathbb{Z}/2)$ is an algebra spectral sequence, with*

$$E^2 \cong \mathbb{Z}/2[t,t^{-1},u_1,x_1,x_2]/(u_1^2=t,x_1^2=0)$$

and $d^2(x_1) = tx_2$, *while* $d^2(u_1) = d^2(t) = d^2(x_2) = 0$. *Thus*

$$E^3 = E^\infty = \mathbb{Z}/2[t, t^{-1}, u_1]/(u_1^2 = t)$$

and $\pi_*(\hat{\mathbb{H}}(C_2, T(\mathbb{F}_2)); \mathbb{Z}/2) \cong \mathbb{Z}/2$ *for all* $*$. *Furthermore*

$$\hat{\Gamma}_1 : T(\mathbb{F}_2) \to \hat{\mathbb{H}}(C_2, T(\mathbb{F}_2))$$

induces an isomorphism on mod two homotopy groups in all nonnegative degrees, whence is a connective two-adic equivalence.

(2) *The spectral sequence* $\hat{E}^*(C_2, \mathbb{F}_2; \mathbb{Z}/4)$ *is an algebra spectral sequence abutting to* $\pi_*(\hat{\mathbb{H}}(C_2, T(\mathbb{F}_2)); \mathbb{Z}/4)$, *with*

$$E^2 = \mathbb{Z}/2[t, t^{-1}, u_1, y_1, y_2]/(u_1^2 = t, y_1^2 = 0)$$

where $d^2 = 0$ *everywhere*, $d^3(u_1) = t^2 y_2$, *and* $d^3(t) = d^3(y_1) = d^3(y_2) = 0$. *Thus*

$$E^4 = E^\infty = \mathbb{Z}/2[t, t^{-1}, y_1]/(y_1^2 = 0).$$

(3) *The spectral sequence* $E^*(C_2, \mathbb{F}_2; \mathbb{Z}/4)$ *abutting to* $\pi_*(T(\mathbb{F}_2)^{hC_2}; \mathbb{Z}/4)$ *has*

$$E^\infty = \mathbb{Z}/2[t, y_1, y_2]/(y_1^2 = 0, t^2 y_2 = 0).$$

So $\pi_*(T(\mathbb{F}_2)^{C_2}; \mathbb{Z}/4)$ *is represented by the classes* $\mathbb{Z}/2[y_1, y_2]/(y_1^2 = 0)\{t, 1\}$ *in nonnegative degrees.*

Proof. The first part is from Section 4 of [11], and the third follows by truncating the results of part (2) to the upper left quadrant. For (2), we compare with the mod two case:

$$\hat{E}^*(C_2, \mathbb{F}_2; \mathbb{Z}/4) \underset{\varepsilon}{\overset{\rho}{\rightleftarrows}} \hat{E}^*(C_2, \mathbb{F}_2; \mathbb{Z}/2)$$

ε is an isomorphism in odd fiber degrees and zero in even fiber degrees, so $d^2 = 0$ from odd fiber degrees on the left. The generator of $\pi_{2n}\hat{\mathbb{H}}(C_2, T(\mathbb{F}_2)) \cong \mathbb{Z}/2$ maps to the permanent cycle t^{-n} on the right, under coefficient reduction, which factors through ρ. Since ρ cannot increase filtration degree, t^{-n} must be an infinite cycle on the left, too, for each integer n.

Each $\pi_{2n}T(\mathbb{F}_2)^{C_2} = \mathbb{Z}/4$ by [11, Theorem 4.5]. So $x_2 \in \pi_2(T(\mathbb{F}_2)^{C_2}; \mathbb{Z}/2)$ is hit from $\pi_2(T(\mathbb{F}_2)^{C_2}; \mathbb{Z}/4)$ under ρ, and the only class that can hit it must stem from y_2, so y_2 survives in $E^*(C_2, \mathbb{F}_2; \mathbb{Z}/4)$. But t^{-1} already survives in this total degree in $\hat{E}^*(C_2, \mathbb{F}_2; \mathbb{Z}/4)$, abutting to $T_2(\mathbb{F}_2; \mathbb{Z}/4) \cong \mathbb{Z}/2$, so y_2 is a boundary there, i.e., $d^3(t^{-2}u_1) = y_2$. Hence $d^2(t^{-2}u_1) = 0$, and since each t^n is an infinite cycle we get $d^2(u_1) = 0$.

$u_1 x_2 \in \pi_1(T(\mathbb{F}_2)^{C_2}; \mathbb{Z}/2)$ is hit from $\pi_1(T(\mathbb{F}_2)^{C_2}; \mathbb{Z}/4)$ by ρ, so either $u_1 y_2$ or y_1 survives. But $d^3(u_1 y_2) = t^2 y_2^2 \neq 0$ by the algebra structure, so it is y_1 which survives in $E^*(C_2, \mathbb{F}_2; \mathbb{Z}/4)$. It cannot be a boundary in $\hat{E}^*(C_2, \mathbb{F}_2; \mathbb{Z}/4)$, so y_1 survives there too. Thus $d^3(u_1) = t^2 y_2$ propagates by the t^n and y_1, to completely determine the latter spectral sequence. $\square$

We can now prove our first main theorem.

Theorem 4.7. *The map Γ_1 is a connective two-adic equivalence. Hence the induced map*

$$\Gamma_1 : T(\mathbb{Z})_2^{C_2} \to T(\mathbb{Z})_2^{hC_2}[0,\infty)$$

is a homotopy equivalence.

Proof. By Lemma 2.3 it suffices to prove that $\hat{\Gamma}_1$ induces an isomorphism in nonnegative degrees from $T_*(\mathbb{Z};\mathbb{Z}/2) \cong \mathbb{Z}/2[e_3,e_4]/(e_3^2=0)$ to $\pi_*(\hat{\mathbb{H}}(C_2,T(\mathbb{Z}));\mathbb{Z}/2)$, which has associated graded groups

$$E^\infty = \mathbb{Z}/2[t^2,t^{-2},e_3]/(e_3^2=0).$$

We use the following commutative square:

$$\begin{array}{ccc} T(\mathbb{Z}) & \xrightarrow{\pi} & T(\mathbb{F}_2) \\ \Big\downarrow{\scriptstyle \hat{\Gamma}_1} & & \Big\downarrow{\scriptstyle \hat{\Gamma}_1} \\ \hat{\mathbb{H}}(C_2,T(\mathbb{Z})) & \xrightarrow{\pi} & \hat{\mathbb{H}}(C_2,T(\mathbb{F}_2)) \end{array}$$

As noted above the right hand vertical map is a connective two-adic equivalence. $e_4^n \in \pi_{4n}(T(\mathbb{Z});\mathbb{Z}/2)$ maps to $x_2^{2n} \in \pi_{4n}(T(\mathbb{F}_2);\mathbb{Z}/2)$ by Theorem 1.1(c) of [4]. And x_2^{2n} maps under $\hat{\Gamma}_1$ to t^{-2n} in $\hat{E}^*(C_2,\mathbb{F}_2;\mathbb{Z}/2)$, modulo terms of lower filtration.

Hence $\hat{\Gamma}_1(e_4^n) \in \pi_{4n}(\hat{\mathbb{H}}(C_2,T(\mathbb{Z}));\mathbb{Z}/2)$ must map to $t^{-2n} \neq 0$ under π and thus is nonzero of filtration $\geq 4n$. The only such class in total degree $4n$ is t^{-2n}, i.e., $\hat{\Gamma}_1(e_4^n) = t^{-2n}$, for all $n \geq 0$.

From Theorem 1.5(2), $\lambda \in K_3(\mathbb{Z})_2$ maps to g_3 in both $T_*(\mathbb{Z})_2$ and $\hat{E}^\infty(C_2)$, whence integrally $\hat{\Gamma}_1(g_3) = g_3$. Using the action of integral homotopy upon mod two homotopy, we get $\hat{\Gamma}_1(e_3 e_4^n) = e_3 t^{-2n}$ for all $n \geq 0$, and the theorem follows. □

Remark 4.8. If we replace $T(\mathbb{Z})$ with $\hat{T}(\mathbb{Z}) = \hat{\mathbb{H}}(C_2,T(\mathbb{Z}))$ in the spectral sequence $\hat{E}^*(C_2;\mathbb{Z}/2)$, we effectively invert e_4. For $\pi_*(\hat{T}(\mathbb{Z});\mathbb{Z}/2) = \mathbb{Z}/2[e_3,e_4,e_4^{-1}]/(e_3^2=0) = T_*(\mathbb{Z};\mathbb{Z}/2)[e_4^{-1}]$. Then all the surviving cycles in $\hat{E}^\infty(C_2;\mathbb{Z}/4)[e_4^{-1}]$ would be hit by differentials crossing the horizontal axis, and the abutment would be zero. So $\hat{\mathbb{H}}(C_2,\hat{T}(\mathbb{Z})) \simeq *$.

Similar remarks apply for the mod four spectral sequence. The permanent cycles in the upper half plane spectral sequence $\hat{E}^*(C_2;\mathbb{Z}/4)$ are precisely the boundaries of differentials crossing the horizontal axis in the extended, full plane spectral sequence $\hat{E}^*(C_2;\mathbb{Z}/4)[f_8^{-1}]$ obtained by inverting f_8. For $\pi_*(\hat{T}(\mathbb{Z});\mathbb{Z}/4) = T_*(\mathbb{Z};\mathbb{Z}/4)[f_8^{-1}]$. This observation provides an alternative way of organizing the present calculation.

5. Short exact sequences of spectral sequences

We continue by establishing Proposition 5.4, which gives a criterion for when a cofiber sequence relating three filtered spectra (left, middle and right) gives rise to short exact sequences of E^r-terms for all r, for the three corresponding spectral sequences. In particular the left spectral sequence will then always inject into the middle one.

Consider a cofiber sequence of spectra

$$X \xrightarrow{i} Y \xrightarrow{j} Z \xrightarrow{\partial} \Sigma X.$$

We assume X, Y and Z are endowed with filtrations $\{X_s\}_s$, $\{Y_s\}_s$ and $\{Z_s\}_s$, where the maps $X_{s-1} \to X_s$, $X_s \to X$ are assumed to be cofibrations, and likewise for Y and Z. We also assume that the maps above respect the filtrations, and induce cofiber sequences

$$(5.1) \qquad X_s \xrightarrow{i_s} Y_s \xrightarrow{j_s} Z_s \xrightarrow{\partial_s} \Sigma X_s.$$

We write $\overline{X}_s = X_s/X_{s-1}$ and likewise for Y and Z, and get cofiber sequences

$$(5.2) \qquad \overline{X}_s \xrightarrow{\bar{\imath}_s} \overline{Y}_s \xrightarrow{\bar{\jmath}_s} \overline{Z}_s \xrightarrow{\bar{\partial}_s} \Sigma \overline{X}_s.$$

Next we assume $\mathrm{hocolim}_{s \to +\infty} X_s \simeq X$ and $\mathrm{holim}_{s \to -\infty} X_s \simeq *$, and similarly for Y and Z. Then associated to these filtered spectra, we have three spectral sequences:

$$E^1_{s,t}(\underline{X}) = \pi_{s+t}(\overline{X}_s) \Rightarrow \pi_{s+t}(X),$$
$$E^1_{s,t}(\underline{Y}) = \pi_{s+t}(\overline{Y}_s) \Rightarrow \pi_{s+t}(Y),$$
$$E^1_{s,t}(\underline{Z}) = \pi_{s+t}(\overline{Z}_s) \Rightarrow \pi_{s+t}(Z).$$

The underline refers to the dependence of the spectral sequence on the filtration on X, Y or Z. Since the filtrations are assumed to be compatible, we get maps of spectral sequences

$$(5.3) \qquad E^r_{s,t}(\underline{X}) \xrightarrow{i_*} E^r_{s,t}(\underline{Y}) \xrightarrow{j_*} E^r_{s,t}(\underline{Z}) \xrightarrow{\partial_*} E^r_{s,t-1}(\underline{X}).$$

The case $r = 1$ of this diagram is part of the long exact sequence in spectrum homotopy associated to the cofiber sequence (5.2). If we assume that $\pi_*(\bar{\partial}_s) = 0$ for all s, then this long exact sequence splits into short exact sequences, and we obtain a short exact sequence of E^1-terms

$$0 \to E^1_{s,t}(\underline{X}) \xrightarrow{i_*} E^1_{s,t}(\underline{Y}) \xrightarrow{j_*} E^1_{s,t}(\underline{Z}) \to 0$$

for all s and t.

In general we cannot expect the later terms ($r \geq 2$) of these spectral sequences to fit into such short exact sequences. We will now give a sufficient criterion for the given cofiber sequence of filtered spectra to induce a short exact sequence of spectral sequences

$$0 \to E^r_{s,t}(\underline{X}) \xrightarrow{i_*} E^r_{s,t}(\underline{Y}) \xrightarrow{j_*} E^r_{s,t}(\underline{Z}) \to 0$$

for all finite r, s and t.

In our cases of interest, the spectral sequences collapse at a finite stage, so the corresponding statements will also hold for the E^∞-terms. We do not discuss the convergence questions that arise when the spectral sequences do not collapse.

Define the maps i^r_s, j^r_s and ∂^r_s by the cofiber sequence

$$X_s/X_{s-r} \xrightarrow{i^r_s} Y_s/Y_{s-r} \xrightarrow{j^r_s} Z_s/Z_{s-r} \xrightarrow{\partial^r_s} \Sigma(X_s/X_{s-r})$$

obtained by comparing (5.1) for s and $s-r$.

Proposition 5.4. *If $\pi_*(\partial^r_s)=0$ for all $r \geq 1$ and s, then*

$$0 \to E^r_{s,t}(\underline{X}) \xrightarrow{i_*} E^r_{s,t}(\underline{Y}) \xrightarrow{j_*} E^r_{s,t}(\underline{Z}) \to 0$$

is a short exact sequence for all finite $r \geq 1$, s and t.

Proof. We begin be recalling how the spectral sequences are constructed. The filtration $\{X_s\}_s$ determines maps $I : X_{s-1} \to X_s$, $J : X_s \to \overline{X}_s$ and $\Delta : \overline{X}_s \to \Sigma X_{s-1}$. There are similar maps for Y and Z. (We denote these maps by capital letters to distinguish them from the filtration preserving maps $i : X \to Y$, $j : Y \to Z$ and $\partial : Z \to \Sigma X$.) Then by definition

$$B^r_{s,t}(\underline{X}) = J_*(\ker I^{r-1}_* : \pi_{s+t}(X_s) \to \pi_{s+t}(X_{s+r-1})),$$
$$Z^r_{s,t}(\underline{X}) = \Delta^{-1}_*(\operatorname{im} I^{r-1}_* : \pi_{s+t-1}(X_{s-r}) \to \pi_{s+t-1}(X_{s-1})).$$

Here $B^r_{s,t}(\underline{X}) \subseteq Z^r_{s,t}(\underline{X}) \subseteq E^1_{s,t}(\underline{X}) = \pi_{s+t}(\overline{X}_s)$, and by definition

$$E^r_{s,t}(\underline{X}) = Z^r_{s,t}(\underline{X})/B^r_{s,t}(\underline{X}).$$

The differential $d^r_{s,t} : E^r_{s,t}(\underline{X}) \to E^r_{s-r,t+r-1}(\underline{X})$ takes the class of a with $\Delta_*(a) = I^{r-1}_*(b)$ to the class of $J_*(b)$. Similarly for Y and Z.

We will prove by induction on r that there are short exact sequences

$$(5.5)\qquad \begin{aligned} &0 \to B^r_{s,t}(\underline{X}) \to B^r_{s,t}(\underline{Y}) \to B^r_{s,t}(\underline{Z}) \to 0,\\ &0 \to Z^r_{s,t}(\underline{X}) \to Z^r_{s,t}(\underline{Y}) \to Z^r_{s,t}(\underline{Z}) \to 0,\\ &0 \to E^r_{s,t}(\underline{X}) \to E^r_{s,t}(\underline{Y}) \to E^r_{s,t}(\underline{Z}) \to 0 \end{aligned}$$

for all $r \geq 1$, and all s and t.

When $r=1$ we have $B^1_{s,t}(\underline{X})=0$ and $Z^1_{s,t}(\underline{X}) = E^1_{s,t}(\underline{X})$, and similarly for Y and Z. By assumption $\pi_*(\overline{\partial}_s) = \pi_*(\partial^1_s) = 0$, so the long exact sequence in homotopy associated to the cofiber sequence (5.2) shows that the sequences in (5.5) are exact for $r=1$.

Assume by induction that the sequences in (5.5) are exact for a given $r \geq 1$. We will show below that the sequence involving Z^{r+1} (omitting the bidegrees and X-, Y- or Z-argument for brevity) is short exact. By the exact sequence

$$0 \to Z^{r+1}/B^r \to E^r \xrightarrow{d^r} B^{r+1}/B^r \to 0$$

and the inductive hypothesis it then follows that the sequence involving B^{r+1} is short exact, using the 3×3-lemma repeatedly. By the exact sequence

$$0 \to B^{r+1} \to Z^{r+1} \to E^{r+1} \to 0$$

it also follows that the sequence involving E^{r+1} is short exact. This will complete the proof of the induction step.

It remains to show exactness of the sequence involving Z^{r+1}. The composite

$$Z^r_s \to E^r_s \xrightarrow{d^r} E^r_{s-r}$$

induces a map from the short exact sequence involving Z^r to the short exact sequence involving E^r, by the inductive hypothesis. The kernel of the composite map is Z^{r+1}_s, so by the snake lemma we have an exact sequence:

$$0 \to Z^{r+1}(\underline{X}) \to Z^{r+1}(\underline{Y}) \to Z^{r+1}(\underline{Z}).$$

We now prove that the right-hand side map is surjective. First compute:

$$\begin{aligned} Z^{r+1}_{s,t}(\underline{X}) &= \Delta_*^{-1}(\operatorname{im} I^r_* : \pi_{s+t-1}(X_{s-r-1}) \to \pi_{s+t-1}(X_{s-1})) \\ &= \Delta_*^{-1} \ker(\pi_{s+t-1}(X_{s-1}) \to \pi_{s+t-1}(X_{s-1}/X_{s-r-1})) \\ &= \ker(\pi_{s+t}(\overline{X}_s) \to \pi_{s+t-1}(X_{s-1}/X_{s-r-1})) \\ &= \operatorname{im}(\pi_{s+t}(X_s/X_{s-r-1}) \to \pi_{s+t}(\overline{X}_s)). \end{aligned}$$

Similarly for Y and Z. Hence there is a commutative diagram

$$\begin{tikzcd} \pi_{s+t}(Y_s/Y_{s-r-1}) \arrow[r, "\pi_*(j^{r+1}_s)"] \arrow[d, two heads] & \pi_{s+t}(Z_s/Z_{s-r-1}) \arrow[r, "\pi_*(\partial^{r+1}_s)"] \arrow[d, two heads] & \pi_{s+t-1}(X_s/X_{s-r-1}) \\ Z^{r+1}_{s,t}(\underline{Y}) \arrow[r] & Z^{r+1}_{s,t}(\underline{Z}) & \end{tikzcd}$$

where the top row is exact, and the vertical maps are surjective. Thus, if $\pi_*(\partial^{r+1}_s) = 0$ it follows that $\pi_*(j^{r+1}_s)$ is surjective, and so the lower map must be surjective too. This completes the proof. □

6. Limiting cases of the Tate spectral sequences

This sections achieves two aims. The first is to study the skeleton spectral sequences for the S^1-Tate construction on $T(\mathbb{Z})$, with coefficients (i) in the mod two Moore spectrum, (ii) in the smash product of two mod two Moore spectra, and (iii) in the suspended mod two Moore spectrum. These are related by a cofiber sequence, and to establish a product structure on the first spectral sequence we use an external pairing landing in the second spectral sequence, and internalize it using the injective map of spectral sequences afforded by Proposition 5.4 of the previous section. This procedure succeeds for the Tate construction on $T(\mathbb{Z})$, as established in Proposition 6.5.

The second aim of this section is to relate the C_{2^n}-Tate constructions to the S^1-Tate construction, by passing either to a limit over the Frobenius maps, or to a colimit over Verschiebung maps. These results substitute for a product structure on the mod two spectral sequence for the C_{2^n}-Tate construction on $T(\mathbb{Z})$, which we are unable to construct directly. The main result in this direction is Proposition 6.7.

We recall Greenlees' doubly infinite filtration of $\tilde{E}S^1$. Here $\tilde{E}S^1$ is defined by the cofiber sequence

$$(6.1)\qquad ES^1_+ \xrightarrow{c} S^0 \to \tilde{E}S^1 \to \Sigma_+ ES^1.$$

(see e.g., Section 2. X_+ denotes X with a disjoint base point added, and by definition $\Sigma_+ X = \Sigma(X_+)$.) As usual we take the unit sphere $S(\mathbb{C}^\infty)$ as a model for ES^1, and the one-point compactification $S^{\mathbb{C}^\infty}$ as a model for $\tilde{E}S^1$. Then ES^1 has the *odd spheres filtration* $\{ES^1_s\}_s$ with

$$ES^1_{2i} = ES^1_{2i+1} = S(\mathbb{C}^i) = S^{2i-1}$$

for all $i \geq 0$.

Greenlees' filtration $\{\tilde{E}S^1_s\}_s$ has

$$\tilde{E}S^1_{2i} = \tilde{E}S^1_{2i+1} = S^{\mathbb{C}^i}$$

for all integers i. When i is negative, this requires a spectrum-level interpretation. The corresponding filtration on the Tate construction

$$\hat{\mathbb{H}}(S^1, T(\mathbb{Z})) = [\tilde{E}S^1 \wedge F(ES^1_+, T(\mathbb{Z}))]^{S^1}$$

may be called the *skeleton filtration*.

The right hand part of the cofiber sequence (6.1) admits the filtration $S^0 \to S^{\mathbb{C}^i} \to \Sigma_+ S(\mathbb{C}^i)$. It induces the homotopy norm-restriction cofiber sequence

$$(6.2)\qquad \Sigma T(\mathbb{Z})_{hS^1} \xrightarrow{N^h} T(\mathbb{Z})^{hS^1} \xrightarrow{R^h} \hat{\mathbb{H}}(S^1, T(\mathbb{Z})) \to \Sigma^2 T(\mathbb{Z})_{hS^1}$$

and corresponding maps of spectral sequences relating the three rightmost terms. Classes mapping nontrivially by N^h are represented in the spectral sequence for $\hat{\mathbb{H}}(S^1, T(\mathbb{Z}))$ by classes supporting differentials which cross the vertical axis. See Theorem 2.15 of [6]. Also see Diagram 10.2 below.

Briefly writing $X = \hat{\mathbb{H}}(S^1, T(\mathbb{Z}))$, the skeleton filtration has $X_{2i} = X_{2i+1} = [S^{\mathbb{C}^i} \wedge F(ES^1_+, T(\mathbb{Z}))]^{S^1}$. Hence the odd filtration layers are trivial, while the even filtration layers are

$$\overline{X}_{2i} = X_{2i}/X_{2i-1} = [\Sigma^{2i-1}(S^1_+) \wedge F(ES^1_+, T(\mathbb{Z}))]^{S^1} \simeq \Sigma^{2i} T(\mathbb{Z})$$

by the Adams isomorphism (see [13, Theorem II.7.1]). Here we are using

$$S^{\mathbb{C}^i}/S^{\mathbb{C}^{i-1}} \cong \Sigma^{2i-1}(S^1_+).$$

So the E^2-term of the spectral sequence is concentrated in even columns, each of which is represented by a copy of $T(\mathbb{Z})$. With mod two coefficients the E^2-term is the S^1-Tate cohomology of $\pi_*(T(\mathbb{Z}); \mathbb{Z}/2)$.

Let $M = S^0/2$ be the mod two Moore spectrum, defined by the cofiber sequence

$$S^0 \xrightarrow{2} S^0 \xrightarrow{i} M \xrightarrow{j} S^1.$$

Smashing with M gives the associated cofiber sequence

$$M \wedge S^0 \xrightarrow{1\wedge 2} M \wedge S^0 \xrightarrow{1\wedge i} M \wedge M \xrightarrow{1\wedge j} M \wedge S^1.$$

Of course $M \wedge S^0 \cong M$ and $M \wedge S^1 \cong \Sigma M$. Recall that $1 \wedge 2$ above factors as

$$M \xrightarrow{j} S^1 \xrightarrow{\eta} S^0 \xrightarrow{i} M$$

(see e.g., [17, Section 1]).

Lemma 6.3. *For any filtration subquotient $X^s_{s-r} = X_s/X_{s-r-1}$ of the skeleton filtration on $\hat{\mathbb{H}}(S^1, T(\mathbb{Z}))$, multiplication by η induces the zero homomorphism on the two-torsion elements*

$$\eta_* : {}_2\pi_*(X^s_{s-r}) \to \pi_{*+1}(X^s_{s-r})/2.$$

Hence the mod two homotopy of each X^s_{s-r} has exponent two, and the filtered cofiber sequence

$$\hat{\mathbb{H}}(S^1, T(\mathbb{Z})) \wedge M \xrightarrow{i} \hat{\mathbb{H}}(S^1, T(\mathbb{Z})) \wedge M \wedge M \xrightarrow{j} \hat{\mathbb{H}}(S^1, T(\mathbb{Z})) \wedge \Sigma M$$

induces a short exact sequence of E^r-terms of spectral sequences, for all r.

Proof. A spectral sequence converging to $\pi_*(X^s_{s-r})$ is clearly given by truncating the skeleton spectral sequence for $\hat{\mathbb{H}}(S^1, T(\mathbb{Z}))$ to filtration degrees $s-r$ through s. The E^2-term of the resulting spectral sequence is concentrated in the even columns, each of which contains a copy of $\pi_* T(\mathbb{Z})$. The only differentials will originate on the horizontal axis, for bidegree reasons. Hence all two-torsion in the abutment $\pi_*(X^s_{s-r})$ sits in (even, odd) bidegrees. Multiplication by η takes such a class to an even total degree, in positive fiber degree, where the abutment is zero. Hence the product is trivial in $\pi_{*+1}(X^s_{s-r})/2$, as claimed. $\square$

Remark 6.4. Note that the corresponding argument for the C_{2^n}-Tate construction fails, although the same conclusion will eventually be seen to hold. Our argument uses the "imparity" of the torsion in $\pi_* T(\mathbb{Z})$ in an essential way, and might not generalize to $T(A)$ for a general ring spectrum (FSP) A. However, for any $\mathbb{Z}$-algebra (ring) A there is an equivariant module action of $T(\mathbb{Z})$ on $T(A)$, which greatly structures the behavior of the spectral sequences computing $\pi_*(T(A)^{hC_{2^n}}; \mathbb{Z}/2)$. This may become useful in the computation of $TC(A)_2$.

We now wish to prove that $\hat{E}^*(S^1; \mathbb{Z}/2)$ is an algebra spectral sequence, in the sense that its differentials are derivations. The E^2-term has the formal algebra structure

$$E^2_{*,*} = \hat{H}^{-*}(S^1, \pi_* T(\mathbb{Z}; \mathbb{Z}/2)) = \mathbb{Z}/2[t, t^{-1}, e_3, e_4]/(e_3^2 = 0).$$

Here $t \in E^2_{-2,0}$, while $e_k \in E^2_{0,k}$ for $k = 3, 4$. (See the discussion following (2.4).)

The pairing $S^0 \wedge M \wedge S^0 \to M$ shows that the spectral sequence above is a left and right module over the integral spectral sequence $\hat{E}^*(S^1)$, which is an algebra spectral sequence, and that the two module actions agree.

Proposition 6.5. *The skeleton spectral sequence $\hat{E}^*(S^1; \mathbb{Z}/2)$ converging to the mod two homotopy of $\hat{\mathbb{H}}(S^1, T(\mathbb{Z}))$ is an algebra spectral sequence, in the sense that its differentials are derivations with respect to the formal algebra structure. By truncation, the same claim holds for the skeleton spectral sequence $E^*(S^1; \mathbb{Z}/2)$ converging to the mod two homotopy of $T(\mathbb{Z})^{hS^1}$.*

This result is somewhat surprising, since the obstruction to splitting the right unit map $1 \wedge i : M \wedge S^0 \to M \wedge M$ factors through η, which maps nontrivially to te_3 in $E^\infty(S^1; \mathbb{Z}/2)$ and to $t^3 e_3 e_4$ in $\hat{E}^\infty(S^1; \mathbb{Z}/2)$. In particular we do not claim that there is an algebra structure on the target $\pi_*(\hat{\mathbb{H}}(S^1, T(\mathbb{Z})); \mathbb{Z}/2)$ of the spectral sequence, nor that it has a multiplicative filtration with associated graded algebra compatible with the E^∞-term of $\hat{E}^*(S^1; \mathbb{Z}/2)$. This is in fact false.

Proof. Inductively assume that $\hat{E}^*(S^1; \mathbb{Z}/2)$ is an algebra spectral sequence up to the E^r-term. On the horizontal axis we may then suppose that the E^r-term consists of the integral powers of some class t^{2^i}. The classes 1, te_4, e_3 and te_3e_4 are hit by 1, $\tilde{\eta}_2$, λ and κ in $K_*(\hat{\mathbb{Z}}_2; \mathbb{Z}/2)$. By the action of the image of $\tilde{v}_4 \in K_4(\hat{\mathbb{Z}}_2; \mathbb{Z}/4)$ as formal multiplication by $t^2 e_4^2$ it follows that all the classes $(te_4)^j$ and $e_3(te_4)^j$ are infinite cycles.

Clearly all differentials in $\hat{E}^*(S^1; \mathbb{Z}/2)$ must be of even length, and originate in (even, even) bidegrees. Any class in an odd fiber degree of $\hat{E}^r(S^1; \mathbb{Z}/2)$ lifts to the integral spectral sequence (by a universal coefficient sequence argument), so any formal product involving it lifts to a natural product formed using one of the pairings $S^0 \wedge M \to M$ or $M \wedge S^0 \to M$. Since the formal product extends these left and right module pairings, any differential on such a formal product is a derivation.

The remaining classes sit in (even, even) bidegrees, and form, by the inductive hypothesis, a free $\mathbb{Z}/2[t^{2^i}, t^{-2^i}]$-module on the powers of te_4. We may assume r is even. Now we use the map of spectral sequences i_* considered above, from the skeleton spectral sequence for $\hat{\mathbb{H}}(S^1, T(\mathbb{Z})) \wedge M$ (i.e., $\hat{E}^*(S^1; \mathbb{Z}/2)$) to the skeleton spectral sequence for $\hat{\mathbb{H}}(S^1, T(\mathbb{Z})) \wedge M \wedge M$. By Lemma 6.3 i_* is injective on all E^r-terms.

Let x and y be nonzero classes of (even, even) bidegrees in the E^r-term of $\hat{E}^*(S^1; \mathbb{Z}/2)$. Then the formal product $x \cdot y$ is nonzero and maps under i_* to the nonzero class in the same bidegree of the $(M \wedge M)$-spectral sequence, which is $x \wedge y$. The exterior pairing satisfies

$$d^r(x \wedge y) = d^r(x) \wedge y + x \wedge d^r(y).$$

By naturality i_* takes $d^r(x \cdot y)$ to the left hand side above. Each of the formal products $d^r(x) \cdot y$ and $x \cdot d^r(y)$ involve a class in odd fiber degrees, so map under i_* to $d^r(x) \wedge y$ and $x \wedge d^r(y)$, respectively. Hence

$$i_*(d^r(x \cdot y)) = i_*(d^r(x) \cdot y) + i_*(x \cdot d^r(y))$$

and so by injectivity of i_*, the proposition follows. □

Next we compare $\hat{\mathbb{H}}(S^1, T(\mathbb{Z}))$ with the homotopy limit of the $\hat{\mathbb{H}}(C_{2^n}, T(\mathbb{Z}))$ over the Frobenius maps, and also compare $\Sigma^{-1}\hat{\mathbb{H}}(S^1, T(\mathbb{Z}))$ with the homotopy colimit of the $\hat{\mathbb{H}}(C_{2^n}, T(\mathbb{Z}))$ over the Verschiebung maps. More precisely we wish to compare the limiting skeleton spectral sequences in mod two homotopy with the skeleton spectral sequence $\hat{E}^*(S^1; \mathbb{Z}/2)$.

For all n the Tate constructions $\hat{\mathbb{H}}(C_{2^n}, T(\mathbb{Z}))$ can be defined by

$$\begin{aligned}\hat{\mathbb{H}}(C_{2^n}, T(\mathbb{Z})) &= [\tilde{E}S^1 \wedge F(ES^1_+, T(\mathbb{Z}))]^{C_{2^n}} \\ &\cong F((S^1/C_{2^n})_+, \tilde{E}S^1 \wedge F(ES^1_+, T(\mathbb{Z})))^{S^1}.\end{aligned}$$

The double covering map $(S^1/C_{2^{n-1}})_+ \to (S^1/C_{2^n})_+$ and the (stable) S^1-equivariant transfer map $(S^1/C_{2^{n-1}})_+ \to (S^1/C_{2^n})_+$ defined the Frobenius and Verschiebung maps below by precomposition:

$$\begin{aligned}&F_n : \hat{\mathbb{H}}(C_{2^n}, T(\mathbb{Z})) \to \hat{\mathbb{H}}(C_{2^{n-1}}, T(\mathbb{Z})), \\ &V_n : \hat{\mathbb{H}}(C_{2^{n-1}}, T(\mathbb{Z})) \to \hat{\mathbb{H}}(C_{2^n}, T(\mathbb{Z})).\end{aligned}$$

The collapse map $(S^1/C_{2^n})_+ \to (S^1/S^1)_+$ and the (stable) S^1-equivariant S^1-transfer map $\Sigma_+(S^1/S^1) \to (S^1/C_{2^n})_+$ likewise define maps for all n:

$$\begin{aligned}&\hat{\mathbb{H}}(S^1, T(\mathbb{Z})) \xrightarrow{F} \hat{\mathbb{H}}(C_{2^n}, T(\mathbb{Z})), \\ &\hat{\mathbb{H}}(C_{2^n}, T(\mathbb{Z})) \xrightarrow{V} \Sigma^{-1}\hat{\mathbb{H}}(S^1, T(\mathbb{Z})).\end{aligned}$$

In the limit we obtain maps

$$F : \hat{\mathbb{H}}(S^1, T(\mathbb{Z})) \to \operatorname*{holim}_{F_n} \hat{\mathbb{H}}(C_{2^n}, T(\mathbb{Z}))$$
$$V : \operatorname*{hocolim}_{V_n} \hat{\mathbb{H}}(C_{2^n}, T(\mathbb{Z})) \to \Sigma^{-1}\hat{\mathbb{H}}(S^1, T(\mathbb{Z})). \tag{6.6}$$

(See [11, Sections 3.2 and 3.3] for further discussion.)

As for $\tilde{E}S^1$, there are doubly infinite Greenlees filtrations of $\tilde{E}C_{2^n}$ for all n. But unlike $\{\tilde{E}S^1_s\}_s$ whose layers were nontrivial only in even filtrations, the Greenlees filtrations of $\tilde{E}C_{2^n}$ have nontrivial layers in every filtration. Some care will be required when comparing the two types of filtrations.

We may still use $ES^1 = S(\mathbb{C}^\infty)$ as our model for EC_{2^n}, by restricting the group action, and similarly use $\tilde{E}S^1 = S^{\mathbb{C}^\infty}$ for $\tilde{E}C_{2^n}$. But while the Greenlees filtration of $\tilde{E}C_{2^n}$ has

$$(\tilde{E}C_{2^n})_{2i} = \tilde{E}S^1_{2i} = S^{\mathbb{C}^i}$$

with the restricted C_{2^n}-action, we let $(\tilde{E}C_{2^n})_{2i+1}$ be obtained from $S^{\mathbb{C}^i}$ by adjoining a C_{2^n}-free $(2i+1)$-cell. This may be arranged so that for all s the sth filtration layer is a copy of $\Sigma^s_+ C_{2^n}$. Clearly this filtration corresponds to the usual two-periodic resolution of the cyclic group (see, e.g., [19, Section V.5]).

This Greenlees filtration for $\tilde{E}C_{2^n}$ gives rise to the mod two spectral sequence $\hat{E}^*(C_{2^n}; \mathbb{Z}/2)$ converging to the mod two homotopy of $\hat{\mathbb{H}}(C_{2^n}; T(\mathbb{Z}))$. The E^2-term is given by the C_{2^n}-Tate cohomology of $\pi_*(T(\mathbb{Z}); \mathbb{Z}/2)$. The Frobenius and Verschiebung maps are defined by acting on $(S^1/C_{2^n})_+$, and hence commute with the filtrations. So there are natural Frobenius and Verschiebung maps connecting the various spectral sequences $\hat{E}^*(C_{2^n}; \mathbb{Z}/2)$ to one another. We describe the relationship to the S^1-spectral sequences in the following proposition.

Proposition 6.7. *The Frobenius maps induce a two-adic homotopy equivalence*

$$F : \hat{\mathbb{H}}(S^1, T(\mathbb{Z})) \to \operatorname*{holim}_{F_n} \hat{\mathbb{H}}(C_{2^n}, T(\mathbb{Z}))$$

and an equivalence of spectral sequences $\hat{E}^*(S^1; \mathbb{Z}/2) \cong \lim_{F_n} \hat{E}^*(C_{2^n}; \mathbb{Z}/2)$. *Likewise the Verschiebung maps induce a two-adic homotopy equivalence*

$$V : \operatorname*{hocolim}_{V_n} \hat{\mathbb{H}}(C_{2^n}, T(\mathbb{Z})) \to \Sigma^{-1}\hat{\mathbb{H}}(S^1, T(\mathbb{Z}))$$

and an equivalence of spectral sequences $\operatorname{colim}_{V_n} \hat{E}^*(C_{2^n}; \mathbb{Z}/2) \cong \Sigma^{-1}\hat{E}^*(S^1; \mathbb{Z}/2)$, *where the desuspension has bidegree* $(-1, 0)$, *i.e., is a left shift.*

Proof. We have a C_{2^n}-equivariant map of filtrations

$$\{(\tilde{E}S^1)_s\}_s \to \{(\tilde{E}C_{2^n})_s\}_s$$

inducing for each n the map of mod two homotopy spectral sequences

$$\hat{E}^*(S^1; \mathbb{Z}/2) \xrightarrow{F} \hat{E}^*(C_{2^n}; \mathbb{Z}/2)$$

compatible with the Frobenius maps F_n. For each n the Frobenius map induces isomorphisms of E^2-terms in even columns, and the zero map in odd columns (since we are taking mod two coefficients). Hence $F_n : \hat{\mathbb{H}}(C_{2^n}, T(\mathbb{Z})) \to \hat{\mathbb{H}}(C_{2^{n-1}}, T(\mathbb{Z}))$ induces an isomorphism on even columns up through the E^r-term, where $r = r_0(n)$ is the length of the first odd differential in $\hat{E}^*(C_{2^n}; \mathbb{Z}/2)$. We know that $r_0(n)$ is monotonically increasing in n (by Proposition 3.8), so likewise $F : \hat{E}^*(S^1; \mathbb{Z}/2) \to \hat{E}^*(C_{2^n}; \mathbb{Z}/2)$ induces isomorphisms on even columns up through the E^r-term with $r = r_0(n)$. Hence the limiting Frobenius map induces an isomorphism of spectral sequences

$$F : \hat{E}^*(S^1; \mathbb{Z}/2) \to \lim_{F_n} \hat{E}^*(C_{2^n}; \mathbb{Z}/2)$$

as claimed.

In the other case, we must shift the Greenlees filtration of $\tilde{E}S^1$ one step to the left. Then we have a C_{2^n}-equivariant map of filtrations

$$\{(\tilde{E}C_{2^n})_s\}_s \to \Sigma^{-1}\{(\tilde{E}S^1)_s\}_s = \{(\tilde{E}S^1)_{s+1}\}_s$$

inducing for each n the map of mod two homotopy spectral sequences

$$\hat{E}^*(C_{2^n}; \mathbb{Z}/2) \xrightarrow{V} \Sigma^{-1}\hat{E}^*(S^1; \mathbb{Z}/2)$$

compatible with the Verschiebung maps. (We discussed compatibility directly before the statement of this proposition.) On the filtration layers the Verschiebung map linking $\hat{\mathbb{H}}(C_{2^{n-1}}, T(\mathbb{Z}))$ to $\hat{\mathbb{H}}(C_{2^n}, T(\mathbb{Z}))$ maps $\Sigma^s T(\mathbb{Z})$ to itself, as the identity for odd s, and as multiplication by two for even s. Hence for each n the Verschiebung map induces isomorphisms of E^2-terms in odd columns, and the zero map in even columns (since we are taking mod two coefficients). Also $V_n : \hat{\mathbb{H}}(C_{2^{n-1}}, T(\mathbb{Z})) \to \hat{\mathbb{H}}(C_{2^n}, T(\mathbb{Z}))$ and $V : \hat{E}^*(C_{2^n}; \mathbb{Z}/2) \to \Sigma^{-1}\hat{E}^*(S^1; \mathbb{Z}/2)$ induce isomorphisms on odd columns up through the E^r-term, where $r = r_0(n)$ as above.

Hence the limiting Verschiebung map induces an isomorphism of spectral sequences

$$V : \operatorname*{colim}_{V_n} \hat{E}^*(C_{2^n}; \mathbb{Z}/2) \to \Sigma^{-1}\hat{E}^*(S^1; \mathbb{Z}/2)$$

as claimed. □

This identification of the limiting form of the even and the odd columns of the spectral sequences $\hat{E}^*(C_{2^n}; \mathbb{Z}/2)$ will serve to replace the multiplicative identification made in the odd primary case, which used the long-surviving classes $u_n \in \hat{H}^1(C_{p^n}; \mathbb{Z}/p)$.

7. The C_4-fixed points of $T(\mathbb{Z})$

We determine the differentials in the Tate spectral sequence $\hat{E}^*(C_4; \mathbb{Z}/2)$, and thus compute $\pi_*(T(\mathbb{Z})^{C_4}; \mathbb{Z}/2)$ up to extensions. The argument is presented so as to generalize to an inductive calculation of $\pi_*(T(\mathbb{Z})^{C_{2^n}}; \mathbb{Z}/2)$ for all $n \geq 2$ by means of Tsalidis' theorem.

The Frobenius and Verschiebung maps F_n and V_n induce isomorphisms of the even and odd columns, respectively, among the E^2-terms of the spectral sequences $\hat{E}^*(C_{2^n};\mathbb{Z}/2)$ for all $n \geq 1$. Recall (from Section 4) the *formal algebra structure* on these E^2-terms:

$$\hat{E}^2(C_{2^n};\mathbb{Z}/2) = \mathbb{Z}/2[t, t^{-1}, u_n, e_3, e_4]/(u_n^2 = 0,\ e_3^2 = 0)$$

with $\deg(t) = (-2, 0)$, $\deg(u_n) = (-1, 0)$, $\deg(e_3) = (0, 3)$ and $\deg(e_4) = (0, 4)$.

The spectral sequence $\hat{E}^*(C_2;\mathbb{Z}/2)$ was determined in Theorem 4.1. Its first nontrivial differentials are the d^4-differentials, which are derivations with respect to this formal algebra structure and determined by $d^4(t^{-1}) = te_3$, $d^4(u_1) = 0$ and the infinite cycles e_3 and te_4. Its d^5-differentials are also derivations, and determined by $d^5(t^{-2}u_1) = te_4$, $d^5(t^2) = 0$ and the infinite cycles just mentioned. Hence we have $r_0(1) = 5$. Finally the E^6-term is concentrated in fiber degrees 3 or less, and the spectral sequence collapses at this stage.

Since the first odd differentials in $\hat{E}^*(C_2;\mathbb{Z}/2)$ were of length five, there are no odd differentials among the $\hat{E}^*(C_{2^n};\mathbb{Z}/2)$ with $n \geq 2$ of length five or less. This follows by Proposition 3.8. Hence as long as $r < 5$ by naturality the Frobenius and Verschiebung maps induce isomorphisms, in even and odd columns respectively, of E^r-terms and d^r-differentials among all the $\hat{E}^*(C_{2^n};\mathbb{Z}/2)$. Therefore each $\hat{E}^*(C_{2^n};\mathbb{Z}/2)$ is an algebra spectral sequence through the (E^4, d^4)-term, with $d^4(t^{-1}) = te_3$ and $d^4(u_n) = 0$. But $d^5 = 0$ in all these later spectral sequences.

The first possible nontrivial odd differential on $\hat{E}^*(C_4;\mathbb{Z}/2)$ has length at least 13. For the first odd differential originates in an odd column by the cited proposition, and $d^r = 0$ if $r \equiv 3 \bmod 4$ by fiber degree considerations. Next $d^r = 0$ on odd columns for $r \equiv 1 \bmod 8$ (i.e., $r = 9$) by bidegree considerations, as the image of any such d^r-differential on a d^4-cycle in $\hat{E}^*(C_4;\mathbb{Z}/2)$ lands in a bidegree completely killed by a d^4-differential. Thus all d^7-, d^9- and d^{11}-differentials are trivial. Hence $r_0(2) \geq 13$. We shall see below that indeed $d^{13} \neq 0$ on $\hat{E}^*(C_4;\mathbb{Z}/2)$, so $r_0(2) = 13$.

Thus up through E^{12} there are no odd differentials in $\hat{E}^*(C_4;\mathbb{Z}/2)$, and again by the cited proposition there are no odd differentials of length 13 or less in $\hat{E}^*(C_{2^n};\mathbb{Z}/2)$ for all $n \geq 3$. Thus the Frobenius and Verschiebung maps F_n and V_n induce isomorphisms up through the E^{13}-terms of all these spectral sequences $\hat{E}^*(C_{2^n};\mathbb{Z}/2)$ with $n \geq 2$.

We now look at the even differentials in this range. First by Proposition 3.11 we have $d^r = 0$ for $r \equiv 2 \bmod 4$ when $r < 13$. Furthermore $d^r = 0$ for $r \equiv 0 \bmod 8$ (i.e., $r = 8$) by bidegree considerations, as again the image of any such d^r-differential on a d^4-cycle in $\hat{E}^*(C_4;\mathbb{Z}/2)$ lands in a bidegree completely killed by the d^4-differentials. Hence the first possible nontrivial even differential after d^4 is d^{12}.

By Proposition 6.7 above, the even columns of each $\hat{E}^r(C_{2^n};\mathbb{Z}/2)$ with $r < r_0(n)$ agree with $\hat{E}^r(S^1;\mathbb{Z}/2)$ through the Frobenius maps, while the odd columns in the same range agree with the left shift $\Sigma^{-1}\hat{E}^r(S^1;\mathbb{Z}/2)$. Hence we can conclude:

Lemma 7.1. *All d^r-differentials in $\hat{E}^*(C_{2^n};\mathbb{Z}/2)$ are derivations for $r < r_0(n)$.*

Proof. By Proposition 6.5 the differentials in $\hat{E}^*(S^1;\mathbb{Z}/2)$ are derivations. Assuming $r<r_0(n)$ we have

$$\hat{E}^r(C_{2^n};\mathbb{Z}/2)\cong\hat{E}^r(S^1;\mathbb{Z}/2)\otimes E(u_n)$$

as algebras, where $E(u_n)$ denotes the exterior algebra on u_n. Since 1 is a permanent cycle, its left shifted image u_n is a d^r-cycle. For d^r to be a derivation, it must satisfy $d^r(x\cdot u_n)=d^r(x)\cdot u_n$ for all x in the even columns, which holds by Proposition 6.7. Since $u_n^2=0$ this is also sufficient for d^r to be a derivation. □

In particular the d^r-differentials of $\hat{E}^*(C_4;\mathbb{Z}/2)$ are derivations for $r\leq 12$. We shall see later that also the d^{13}-differentials are derivations, by a more indirect argument.

By truncating $\hat{E}^*(C_2;\mathbb{Z}/2)$ to the upper left quadrant we obtain the spectral sequence $E^*(C_2;\mathbb{Z}/2)$ abutting to the mod two homotopy of $T(\mathbb{Z})^{hC_2}$. On connective covers this agrees with the mod two homotopy of $T(\mathbb{Z})^{C_2}$, by Theorem 0.2. The E^∞-term has additive generators

$$1,\ tu_1e_3;\ te_3;\ te_4;\ e_3;\ t^2e_4^2;\ te_3e_4;\ u_1e_3e_4;\ e_3e_4,\ t^2e_3e_4^2,$$

in total degrees zero through seven, together with their formal multiples with powers of e_4^2. Hence we have the following calculation (in nonnegative degrees),

$$(7.2)\qquad \pi_*(T(\mathbb{Z})^{C_2};\mathbb{Z}/2)\cong\begin{cases}(\mathbb{Z}/2)^2 & \text{when } *\equiv 0,7 \bmod 8,\\ \mathbb{Z}/2 & \text{otherwise.}\end{cases}$$

(We shall see in Remark 8.15 that this is the correct additive structure, i.e., that the groups have exponent two. We shall only need the order of these groups in our arguments, so we omit the easy proof.)

By Tsalidis' theorem [20], the map $\hat{\Gamma}_n: T(\mathbb{Z})^{C_{2^{n-1}}}\to\hat{\mathbb{H}}(C_{2^n},T(\mathbb{Z}))$ induces a two-adic equivalence on connective covers for all $n\geq 1$. Hence $\hat{E}^*(C_4;\mathbb{Z}/2)$ abuts to the groups listed above, at least in nonnegative degrees.

Now consider the action of the mod four spectral sequences upon the mod two spectral sequences. We recall from [17], or Theorem 1.6, the generators $f_n\in\pi_n(T(\mathbb{Z});\mathbb{Z}/4)$ for $n=3,4,7$ and 8. By Theorem 1.9, the class $\tilde{v}_4\in K_4(\hat{\mathbb{Z}}_2;\mathbb{Z}/4)$ maps to t^2f_8 in the mod four spectral sequence $E^*(S^1;\mathbb{Z}/4)$, which in turn reduces to $(te_4)^2$ in the mod two spectral sequence $E^*(S^1;\mathbb{Z}/2)$.

Lemma 7.3. *The class $\tilde{v}_4\mapsto t^2f_8\in\hat{E}^2_{-4,8}(C_{2^n};\mathbb{Z}/4)$ is a permanent cycle for all $n\geq 2$. Hence multiplication by t^2f_8 acts as formal multiplication by the permanent cycle $(te_4)^2$ on $\hat{E}^*(C_{2^n};\mathbb{Z}/2)$ for all $n\geq 2$. In particular*

$$d^r((te_4)^2\cdot x)=(te_4)^2\cdot d^r(x)$$

for all x and r.

Thus all differentials in $\hat{E}^*(C_{2^n};\mathbb{Z}/2)$ propagate up and to the left by formal multiplication by $(te_4)^2$. The integral permanent cycle representing λ likewise propagates differentials vertically by formal multiplication by e_3.

Proof. Consider the spectral sequences $\hat{E}^*(C_2;\mathbb{Z}/4)$ and $\hat{E}^*(C_4;\mathbb{Z}/4)$, and the maps F_1 and V_1 connecting them. In the latter sequence $t^2 f_8$ is not a d^2-boundary since tf_7 is an infinite cycle (it is hit by κ), not a d^5-boundary since $d^5(t^{-1}u_2 f_4) = V_1(d^5(t^{-1}u_1 f_4)) = 0$, and not a d^6-boundary since $t^{-1}f_3 = d^4(t^{-3})$. The d^9-differential landing in bidegree $(-4,8)$ originates in $\hat{E}^9_{5,0}(C_4;\mathbb{Z}/4)$, which is contained in $\hat{E}^5_{5,0}(C_4;\mathbb{Z}/4) = \mathbb{Z}/2\{2t^{-2}u'_2\}$, in light of $d^4(t^{-2}u'_2) = u_2 f_3$. (In fact $d^8(t^{-3}u'_2) = \varepsilon(d^8(t^{-3}u'_1)) = \varepsilon(0) = 0$ so $E^5 = E^9$ in this bidegree.) Then $d^9(2t^{-3}u'_2) = V_1(d^9(t^{-3}u'_1)) = V_1(t^2 f_8) = 2t^2 f_8$, and we see that $t^2 f_8$ itself is not in the image of d^9. Thus $t^2 f_8$ is a permanent cycle in $\hat{E}^*(C_4;\mathbb{Z}/4)$. A similar analysis using the V_n for $n \geq 2$ shows that $t^2 f_8$ is not a boundary and thus survives to E^∞ in all $\hat{E}^*(C_{2^n};\mathbb{Z}/4)$ with $n \geq 2$. □

The first d^{12}-differential arises as follows.

Lemma 7.4. *In the spectral sequence $\hat{E}^*(C_4;\mathbb{Z}/2)$ the classes $t^{2i}\cdot e_3$ and $t^{2i}\cdot te_3e_4$ are permanent cycles for all integers i. There is a differential*

$$d^{12}(t^{-2}) = t^4 e_3 e_4^2.$$

Proof. By comparison with $\hat{E}^*(S^1)$, all classes in (even, odd) bidegrees are infinite cycles. Among these the classes $t^{2i}e_3$ and $t^{2i+1}e_3e_4$ are not d^4-boundaries, and thus survive to E^∞. Since $\pi_*(T(\mathbb{Z})^{C_2};\mathbb{Z}/2) \cong \mathbb{Z}/2$ in (positive) degrees $* \equiv 1,3,5 \bmod 8$ there can be no further permanent cycles in these total degrees. Thus all other infinite cycles in these total degrees are boundaries. An early case of this is the class $t^4 e_3 e_4^2$ in fiber degree 11, which can only be hit by the given d^{12}-differential. □

Lemma 7.5. *In the spectral sequence $\hat{E}^*(S^1;\mathbb{Z}/2)$ the d^r-differentials for $r \leq 12$ are determined by being derivations, the formulas $d^4(t^{-1}) = te_3$ and $d^{12}(t^{-2}) = t^4 e_3 e_4^2$, and that e_3 and te_4 are permanent cycles.*

Hence each spectral sequence $\hat{E}^*(C_{2^n};\mathbb{Z}/2)$ for $n \geq 2$ is an algebra spectral sequence through the (E^{12}, d^{12})-term, with differentials determined by $d^4(t^{-1}) = te_3$, $d^{12}(t^{-2}) = t^4 e_3 e_4^2$, $d^{12}(u_n) = 0$ and that e_3 and te_4 are permanent cycles.

Proof. The d^{12}-differential just found in $E^*(C_4;\mathbb{Z}/2)$ translates to the S^1-spectral sequence by the Frobenius map, and generates the remaining d^{12}-differentials there by the algebra structure. (In particular $d^{12}(te_4 \cdot t^{-2}) \neq 0$. It appears to be very difficult to prove this within the C_4-spectral sequence without having established the algebra structure on the S^1-spectral sequence.)

Thus the structure of the C_{2^n}-spectral sequences up to the E^{13}-terms follows from Proposition 6.7, and is as claimed. □

The last step of the induction is to determine the d^{13}-differentials in $\hat{E}^*(C_4;\mathbb{Z}/2)$.

Lemma 7.6. *In the spectral sequence $\hat{E}^*(C_4;\mathbb{Z}/2)$ there are differentials*

$$d^{13}(t^{4i}\cdot t^{-4}u_2e_3)=t^{4i}\cdot e_3(te_4)^3,$$
$$d^{13}(t^{4i}\cdot t^{-3}u_2e_3e_4)=t^{4i}\cdot e_3(te_4)^4$$

for all integers i.

Proof. Consider first the classes $(te_4)^3e_3$ and $(te_4)^4e_3$ in $\hat{E}^*(C_4;\mathbb{Z}/2)$. They are in (even, odd) bidegrees; thus they are infinite cycles. They are in total degrees 9 and 11, where by the inductive input (7.2) the abutment has order two. But by Lemma 7.4, the permanent cycles $t^{-2}e_3$ and $t^{-1}e_3e_4$ are already known to generate the abutment in these total degrees. Hence there can be no more permanent cycles in these degrees, and the two infinite cycles considered must be boundaries. Inspecting the E^{13}-term, the only classes that can support differentials hitting $(te_4)^3e_3$ and $(te_4)^4e_3$ are $t^{-4}u_2e_3$ and $t^{-3}u_2e_3e_4$ respectively. Hence we must have the claimed differentials, both of which have length 13.

The same argument in total degrees $(9-8i)$ and $(11-8i)$ for any integer i, i.e., formally multiplying each class mentioned by t^{4i}, implies the general case. □

Note that a separate counting argument is made in each case. There is no obvious topological action available which induces multiplication by powers of t^4. The precise behavior in negative total degrees might be deemed troublesome, but note that this never affects the results in positive degrees. In fact the C_{2^n}-equivariant James periodicity of $\tilde{E}S^1$ proves that the differential structure in positive total degrees also propagates periodically across the upper left quadrant, as desired. The a priori period for the James periodicity is significantly longer than that realized in the spectral sequences under consideration, but this does not affect the argument.

Lemma 7.7. *In the spectral sequence $\hat{E}^*(C_4;\mathbb{Z}/2)$ the d^{13}-differentials are determined by being derivations, the formula $d^{13}(t^{-4}u_2)=(te_4)^3$, and that e_3, te_4 and t^4 are permanent cycles. The resulting E^{14}-term is concentrated in fiber degrees 11 or less, and so the spectral sequence collapses at this stage.*

Proof. From the formal action by e_3 (induced by λ) we conclude that there are differentials

$$d^{13}(t^{4i}\cdot t^{-4}u_2)=t^{4i}\cdot (te_4)^3,$$
$$d^{13}(t^{4i}\cdot t^{-3}u_2e_4)=t^{4i}\cdot (te_4)^4$$

for all integers i. By the formal action of e_3 and $(te_4)^2$ provided by Lemma 7.3 (induced by λ and $\tilde{v}_4$), this determines all the d^{13}-differentials of $\hat{E}^*(C_4;\mathbb{Z}/2)$. By bookkeeping the classes surviving to E^{14} are

$$1,\ t^3u_2e_3e_4;\ t^3e_3e_4;\ te_4;\ e_3;\ t^2e_4^2;\ te_3e_4;\ t^{-2}u_2e_3;\ t^{-2}e_3,\ t^2e_3e_4^2$$

with semicolons separating the degrees, together with all their translates by integral powers of t^4. All these classes have fiber degree 11 or less, so $E^{14} = E^{\infty}$. In particular t^4 is a permanent cycle, as claimed. $\square$

The two classes in total degree zero represent the mod two reductions of generators of $\pi_0 T(\mathbb{Z})^{C_2} \cong \mathbb{Z}^2$. In higher degrees $\eta \mapsto t^3 e_3 e_4$, $\tilde{\eta}_2 \mapsto te_4$, $\lambda \mapsto e_3$, $\rho\tilde{v}_4 \mapsto t^2 e_4^2 = (te_4)^2$, $\kappa \mapsto te_3 e_4$ and $\bar{\sigma} \mapsto t^2 e_3 e_4^2$ generate the image from $K_*(\hat{\mathbb{Z}}_2; \mathbb{Z}/2)$.

We have proved:

Lemma 7.8. *The spectral sequence $\hat{E}^*(C_4; \mathbb{Z}/2)$ is an algebra spectral sequence in the sense that its differentials are derivations with respect to the formal algebra structure on the E^2-term. The differentials are determined by the formulas $d^4(t^{-1}) = te_3$, $d^{12}(t^{-2}) = t^4 e_3 e_4^2$ and $d^{13}(t^{-4} u_2) = (te_4)^3$, and by the permanent cycles e_3, te_4 and t^4.*

The spectral sequence abuts additively to $\pi_*(\hat{\mathbb{H}}(C_4, T(\mathbb{Z})); \mathbb{Z}/2)$. It collapses at the E^{14}-term, with abutment the free $\mathbb{Z}/2[t^4, t^{-4}]$-module on the following generators:

$$1,\ t^3 u_2 e_3 e_4;\ t^3 e_3 e_4;\ te_4;\ e_3;\ t^2 e_4^2;\ te_3 e_4;\ t^{-2} u_2 e_3;\ t^{-2} e_3,\ t^2 e_3 e_4^2.$$

By truncating to the upper left quadrant we find the following.

Lemma 7.9. *The spectral sequence $E^*(C_4; \mathbb{Z}/2)$ is an algebra spectral sequence in the sense that its differentials are derivations with respect to the formal algebra structure on the E^2-term. The differentials are determined by the formulas $d^4(t) = t^3 e_3$, $d^{12}(t^2) = t^8 e_3 e_4^2$ and $d^{13}(u_2) = t^7 e_4^3$, and by the permanent cycles e_3, te_4 and t^4.*

In nonnegative degrees the resulting E^{∞}-term has three permanent cycles in total degree $* \equiv 0, 15 \bmod 16$ and two permanent cycles otherwise. Hence for $* \geq 0$ the mod two homotopy groups of $T(\mathbb{Z})^{C_4}$ have orders

$$\#\pi_*(T(\mathbb{Z})^{C_4}; \mathbb{Z}/2) = \begin{cases} 2^3 & \text{when } * \equiv 0, 15 \bmod 16, \\ 2^2 & \text{otherwise.} \end{cases}$$

Proof. The first part follows from the result for $\hat{E}^*(C_4; \mathbb{Z}/2)$ by naturality and truncation. The second part holds because $\Gamma_2 : T(\mathbb{Z})^{C_4} \to T(\mathbb{Z})^{hC_4}$ is a two-adic connective equivalence by Tsalidis' theorem. The bookkeeping to identify the permanent cycles in the truncated sequence can easily be done by hand here. (The general case will be presented in Section 9 below.) $\square$

Lemmas 7.8 and 7.9 constitute the inductive hypothesis in the case $n = 2$ for the general inductive argument, presented in the next section.

8. The induction argument

We impose the formal algebra structure on the E^2-term

$$\hat{E}^2(C_{2^n}; \mathbb{Z}/2) = \mathbb{Z}/2[t, t^{-1}, u_n, e_3, e_4]/(u_2^2 = 0, e_3^2 = 0)$$

of each spectral sequence $\hat{E}^*(C_{2^n};\mathbb{Z}/2)$. Let

$$\rho(n)=4(2^n-1)$$

for all n. We make the following inductive hypothesis for a given $n\geq 1$:

Hypothesis 8.1. *In the spectral sequence $\hat{E}^*(C_{2^n};\mathbb{Z}/2)$ there are differentials*

$$d^r(t^{-2^k})=t^{2^k}\cdot e_3\cdot(te_4)^{2(2^k-1)}$$

with $r=\rho(k+1)$ for all $0\leq k<n$, and

$$d^r(t^{-2^n}\cdot u_n)=(te_4)^{2^n-1}$$

with $r=\rho(n)+1$. All d^r-differentials for $r\leq\rho(n)+1$ are derivations with respect to the formal algebra structure, and the classes e_3, te_4 and $t^{\pm 2^n}$ survive to the $E^{\rho(n)+2}$-term.

Hence, in the notation of Definition 3.7, the shortest odd differential in $\hat{E}^*(C_{2^n};\mathbb{Z}/2)$ has length $r_0(n)=\rho(n)+1$, at least for the given n. We write $P_h(x)=\mathbb{Z}/2[x]/(x^h=0)$ for the truncated polynomial algebra on x of height h. Let $v_2(i)$ be the 2-adic valuation of i.

Lemma 8.2. *The differentials given in the inductive hypothesis above determine the remaining differentials in $\hat{E}^*(C_{2^n};\mathbb{Z}/2)$, and the spectral sequence collapses at the $E^{\rho(n)+2}$-term. The E^∞-term is*

$$\hat{E}^\infty(C_{2^n};\mathbb{Z}/2)=P_{2^n-1}(te_4)[t^{2^n},t^{-2^n}]\{1,e_3\}\oplus\bigoplus_{k=v_2(i)<n}P_{2(2^k-1)}(te_4)\cdot t^ie_3\{1,u_n\}.$$

Hence all differentials in this spectral sequence are derivations, and the classes e_3, te_4 and $t^{\pm 2^n}$ are permanent cycles.

Proof. For $r<\rho(n)+1$ there are no odd d^r-differentials, so in this range each $\hat{E}^r(C_{2^n};\mathbb{Z}/2)$ is the exterior algebra on u_n over the even columns. Consider the even columns of the E^2-term as a free $\mathbb{Z}/2[te_4]$-module on the generators t^i and t^ie_3. There is a d^r-differential on t^i precisely when $v_2(i)=k<n$ and $r=\rho(k+1)$, which hits $t^{2^{k+1}+i}\cdot e_3\cdot(te_4)^{2(2^k-1)}$. The even differentials on t^ie_3 are zero for bidegree reasons. Hence the classes surviving to the even columns of the $E^{\rho(n)+1}$-term are of two kinds: those in (even, even) bidegrees which have not supported a differential, and those in (even, odd) bidegrees which were not hit by a differential. The first classes form the free $\mathbb{Z}/2[te_4]$-module on the generators t^i with $v_2(i)\geq n$, i.e., on the integral powers of t^{2^n}. The second classes form the sum of the polynomial algebras in te_4 truncated at height $2(2^k-1)$ on the generators t^ie_3, where $k=v_2(i)$. If $v_2(i)\geq n$ there is no truncation.

Hence

(8.3) $$\hat{E}^{\rho(n)+1}_{ev,*}(C_{2^n};\mathbb{Z}/2)=\mathbb{Z}/2[te_4,t^{2^n},t^{-2^n}]\{1,e_3\}\oplus\bigoplus_{k=v_2(i)<n}P_{2(2^k-1)}(te_4)\cdot t^ie_3,$$

$$\hat{E}^{\rho(n)+1}_{*,*}(C_{2^n};\mathbb{Z}/2)=\hat{E}^{\rho(n)+1}_{ev,*}(C_{2^n};\mathbb{Z}/2)\otimes E(u_n).$$

Now consider the odd differential generated by

$$d^r(t^iu_n)=t^{2^n+i}(te_4)^{2^n-1}$$

whenever $v_2(i)\geq n$, where $r=\rho(n)+1$. It maps the odd columns to the even columns, and is trivial on the even columns. On u_n times the first (polynomial) summands above, the differential is injective, with cokernel the free $P_{2^n-1}(te_4)$-module on generators t^i and t^ie_3 with $v_2(i)\geq n$. On u_n times the second (truncated) summands above, the differential is zero because $2^n-1>2(2^k-1)$ for all $k<n$. Hence

$$\hat{E}^{\rho(n)+2}(C_{2^n};\mathbb{Z}/2)=P_{2^n-1}(te_4)[t^{2^n},t^{-2^n}]\{1,e_3\}\oplus\bigoplus_{k=v_2(i)<n}P_{2(2^k-1)}(te_4)\cdot t^ie_3\{1,u_n\}.$$

This E^r-term is concentrated in fiber degrees $4(2^n-2)+3$ or less, and since $\rho(n)>4(2^n-2)+3$ there can be no further differentials, at this stage or later. Hence the spectral sequence collapses as claimed. □

Lemma 8.4. *The spectral sequence $E^*(C_{2^n};\mathbb{Z}/2)$ has E^2-term*

$$E^2(C_{2^n};\mathbb{Z}/2)=\mathbb{Z}/2[t,u_n,e_3,e_4]/(u_n^2=0,e_3^2=0)$$

and differentials

$$d^r(t^{2^k})=t^{3\cdot 2^k}\cdot e_3\cdot(te_4)^{2(2^k-1)}$$

with $r=\rho(k+1)$ for all $0\leq k<n$, and

$$d^r(u_n)=t^{2^n}\cdot(te_4)^{2^n-1}$$

with $r=\rho(n)+1$. All differentials are derivations with respect to the formal algebra structure, and the classes e_3, te_4 and t^{2^n} are permanent cycles. This determines all the remaining differentials. The $E^{\rho(n)+2}$-term of the spectral sequence is concentrated in fiber degree $4(2^n-2)+3$ or less, and so the spectral sequence collapses at this stage. The E^∞-term satisfies

$$E^\infty(C_{2^n};\mathbb{Z}/2)=P_{2^{n+1}-1}(te_4)[e_4^{2^n},e_4^{-2^n}]\{1,e_3\}\oplus\bigoplus_{k=v_2(i)<n}P_{2(2^{k+1}-1)}(te_4)\cdot e_3e_4^i\{1,u_n\}$$

in nonnegative total degrees.

Proof. All claims are clear by comparison over the homotopy restriction map

$$E^*R_n^h:E^*(C_{2^n};\mathbb{Z}/2)\to\hat{E}^*(C_{2^n};\mathbb{Z}/2),$$

except perhaps the calculation of the E^∞-term. We compare the spectral sequence with its localization where e_4 is inverted, at the E^2-term. These two spectral sequences agree in nonnegative total degrees.

For $r<\rho(n)+1$ there are no odd d^r-differentials, so in this range each term $E^r(C_{2^n};\mathbb{Z}/2)$ is the exterior algebra on u_n over the even columns. At the E^2-term, these even columns agree in nonnegative total degrees with the free $\mathbb{Z}/2[te_4]$-module on the generators e_4^i and $e_3e_4^i$, with $i\in\mathbb{Z}$. There is a d^r-differential on e_4^i precisely when $v_2(i)=k<n$ and $r=\rho(k+1)$, which hits $e_4^{i-2^{k+1}}\cdot e_3\cdot(te_4)^{2(2^{k+1}-1)}$. Hence the classes surviving in the even columns of the $E^{\rho(n)+1}$-term are again of two kinds: those in (even, even) bidegrees which have not supported a differential, and those in (even, odd) bidegrees which were not hit by a differential. The first classes form the free $\mathbb{Z}/2[te_4]$-module on the generators e_4^i with $v_2(i)\geq n$, i.e., the integral powers of $e_4^{2^n}$. The second classes form the sum of the truncated polynomial algebras in te_4 of height $2(2^{k+1}-1)$ on the generators $e_3e_4^i$, where $k=v_2(i)$. If $k\geq n$ there is no truncation. Hence

$$E^{\rho(n)+1}_{ev,*}(C_{2^n};\mathbb{Z}/2)=\mathbb{Z}/2[te_4,e_4^{2^n},e_4^{-2^n}]\{1,e_3\}\oplus\bigoplus_{k=v_2(i)<n}P_{2(2^{k+1}-1)}(te_4)\cdot e_3e_4^i$$

in nonnegative total degrees. Now consider the odd differential generated by

$$d^r(e_4^iu_n)=(te_4)^{2^{n+1}-1}\cdot e_4^{i-2^n}$$

whenever $v_2(i)\geq n$, where $r=\rho(n)+1$. It is injective on u_n times the first (polynomial) summands above, with cokernel the free $P_{2^{n+1}-1}(te_4)$-module on generators e_4^i and $e_3e_4^i$ with $v_2(i)\geq n$. On u_n times the second (truncated polynomial) summands above it is zero, because $2^{n+1}-1>2(2^{k+1}-1)$ for all $k<n$. Hence

$$\begin{aligned}&E^{\rho(n)+2}(C_{2^n};\mathbb{Z}/2)\\&\quad=P_{2^{n+1}-1}(te_4)[e_4^{2^n},e_4^{-2^n}]\{1,e_3\}\oplus\bigoplus_{k=v_2(i)<n}P_{2(2^{k+1}-1)}(te_4)\cdot e_3e_4^i\{1,u_n\}\end{aligned}$$

in nonnegative total degrees, as claimed. □

Lemma 8.5. *There are n permanent cycles in each nonnegative total degree of $E^*(C_{2^n};\mathbb{Z}/2)$, except in the degrees congruent to -1 or $0 \bmod 2^{n+2}$ where there are $(n+1)$ permanent cycles.*

Proof. The first part

$$P_{2^{n+1}-1}(te_4)[e_4^{2^n},e_4^{-2^n}]\{1,e_3\}$$

of the E^∞-term has one generator in each total degree except the degrees -2 and $1 \bmod 2^{n+2}$.

For a fixed $k\geq 0$, the sum of the terms

$$P_{2(2^{k+1}-1)}(te_4)\cdot e_3e_4^i\{1,u_n\}$$

with $v_2(i)=k$ contains one generator in each total degree, except in the degrees $2^{k+2}-2$, $2^{k+2}-1$, 2^{k+2} and $2^{k+2}+1 \bmod 2^{k+3}$. Summing over $0 \le k < n$ we find that the second part of the E^{∞}-term has $(n-1)$ generators in each degree, except in the degrees -2, -1, 0 and $1 \bmod 2^{n+2}$, where there are n generators.

Hence the E^{∞}-term itself has rank n in every total degree, except in the degrees -1 and $0 \bmod 2^{n+2}$, where the rank is $(n+1)$. □

The induction step is made possible by the following result.

Lemma 8.6. *In nonnegative total degrees the spectral sequence $\hat{E}^*(C_{2^{n+1}};\mathbb{Z}/2)$ has n permanent cycles in every total degree, except in the degrees congruent to -1 or $0 \bmod 2^{n+2}$, where there are $(n+1)$ permanent cycles.*

Proof. Tsalidis' theorem from [20] and the initial input Theorem 0.2 ensure that there is a two-adic equivalence of connective covers induced by the map

$$\hat{\Gamma}_{n+1} : T(\mathbb{Z})^{C_{2^n}} \to \hat{\mathbb{H}}(C_{2^{n+1}}, T(\mathbb{Z})).$$

Hence in nonnegative total degrees the order of the abutment of $E^*(C_{2^n};\mathbb{Z}/2)$ equals the order of the abutment of $\hat{E}^*(C_{2^{n+1}};\mathbb{Z}/2)$. The claim follows. □

Lemma 8.7. *The even columns of the $E^{\rho(n)+1}$-term of $\hat{E}^*(C_{2^{n+1}};\mathbb{Z}/2)$ are given by*

$$\hat{E}^{\rho(n)+1}_{ev,*}(C_{2^{n+1}};\mathbb{Z}/2) = \mathbb{Z}/2[te_4, t^{2^n}, t^{-2^n}]\{1, e_3\} \oplus \bigoplus_{k=v_2(i)<n} P_{2(2^k-1)}(te_4)\cdot t^i e_3$$

and the full spectral sequence satisfies

$$\hat{E}^{\rho(n)+1}_{*,*}(C_{2^{n+1}};\mathbb{Z}/2) = \hat{E}^{\rho(n)+1}_{ev,*}(C_{2^{n+1}};\mathbb{Z}/2) \otimes E(u_{n+1}).$$

The d^r-differential for $r=\rho(n)+1$ is zero.

Proof. Up to the first odd differential in $\hat{E}^*(C_{2^n};\mathbb{Z}/2)$, of length $r_0(n)=\rho(n)+1$, the Frobenius and Verschiebung maps induce isomorphisms of E^r-terms and d^r-differentials between $\hat{E}^*(C_{2^n};\mathbb{Z}/2)$ and $\hat{E}^*(C_{2^{n+1}};\mathbb{Z}/2)$, taking u_n to u_{n+1} in the odd columns. This explains the formulas above, in view of (8.3). By Proposition 3.8 the first odd differential of $\hat{E}^*(C_{2^{n+1}};\mathbb{Z}/2)$ is strictly longer than that of $\hat{E}^*(C_{2^n};\mathbb{Z}/2)$, which had length $\rho(n)+1$. □

Lemma 8.8. *For $\rho(n)<r<\rho(n+1)$ the d^r-differentials in $\hat{E}^*(C_{2^{n+1}};\mathbb{Z}/2)$ are zero.*

Proof. The permanent cycles $t^2 f_8$ and f_3 (coming from $\tilde{v}_4$ and λ in $K_*(\hat{\mathbb{Z}}_2;\mathbb{Z}/4)$) act upon the spectral sequence by formal multiplication by $(te_4)^2$ and e_3. Each class in the given $E^{\rho(n)+1}$-term can be written as the product of a class in fiber degree 0 or 4 with a power of $(te_4)^2$ and e_3. Thus it suffices to show that there are no d^r-differentials originating in fiber degree 0 or 4 when r is in the stated range. The only classes surviving in these fiber degrees are t^i and $t^i u_{n+1}$ with $v_2(i) \geq n$, and te_4 times these classes.

When $k < n$ the summand $P_{2(2^k-1)}(te_4) \cdot t^i e_3\{1, u_n\}$ in the given $E^{\rho(n)+1}$-term is concentrated in fiber degrees 3 through $4(2(2^k-1)-1)+3 < \rho(n)$. Hence none of the d^r-differentials with $r > \rho(n)$ can land in these summands. The remaining summands constitute the free $\mathbb{Z}/2[te_4, e_3]/(e_3^2 = 0)$-module on the classes t^i and $t^i u_{n+1}$ with $v_2(i) \geq n$.

We have thus identified the relevant possible sources and targets of any nontrivial d^r-differentials with $r > \rho(n)$. Bidegree considerations will now show that any such differentials are zero as long as $r < \rho(n+1)$.

A d^r-differential on t^{-2^n} lands in total degree $2^{n+1} - 1$. The possible targets in this total degree are $(te_4)^{2^n-2} e_3$ and $t^{-2^n} u_{n+1}$, along with their multiples with powers of $(t^2 e_4)^{2^n}$. The latter family (involving u_{n+1}) is excluded by an S^1-comparison argument: there can only be even differentials originating from the even columns. The former family lives in fiber degrees $-5 \bmod 2^{n+2}$ and can only equal $d^r(t^{-2^n})$ for $r \equiv -4 \bmod 2^{n+2}$. Clearly there are no such r *strictly* between $\rho(n) = 2^{n+2}-4$ and $\rho(n+1) = 2^{n+3}-4$.

A d^r-differential on $t^{-2^n} u_{n+1}$ lands in total degree $2^{n+1} - 2$ where the possible targets are $(te_4)^{2^n-2} u_{n+1} e_3$ and $(te_4)^{2^n-1}$, and their multiples with powers of $(t^2 e_4)^{2^n}$. These classes live in fiber degrees -5 and $-4 \bmod 2^{n+2}$, and would be hit by d^r-differentials with $r \equiv -4$ or $-3 \bmod 2^{n+2}$. The former cases fall just outside the range considered, as above, while there is one bidegree where we could have a differential of the latter type, namely $r = \rho(n) + 1$. But this is precisely the differential present in $\hat{E}^*(C_{2^n}; \mathbb{Z}/2)$ but excluded in $\hat{E}^*(C_{2^{n+1}}; \mathbb{Z}/2)$ by the last clause of Lemma 8.7.

Similar arguments applies to all t^i and $t^i u_{n+1}$ with $v_2(i) \geq n$, and to their products with te_4, and the lemma follows. □

Corollary 8.9. *Any nontrivial d^r-differential in $\hat{E}^*(C_{2^{n+1}}; \mathbb{Z}/2)$ with $r > \rho(n)$ has $r \geq \rho(n+1)$.*

The first possible nontrivial odd differential in $\hat{E}^*(C_{2^{n+1}}; \mathbb{Z}/2)$ has length at least $\rho(n+1)+1$. So $r_0(n+1) \geq \rho(n+1)+1$.

We now know that the E^r-terms of $\hat{E}^*(C_{2^n}; \mathbb{Z}/2)$ and $\hat{E}^*(C_{2^{n+1}}; \mathbb{Z}/2)$ are isomorphic through the Frobenius and Verschiebung maps as long as $r \leq \rho(n) + 1$, while the $d^{\rho(n)+1}$-differentials present in the former spectral sequence are zero in the latter. By the corollary above the next possible differential in the latter spectral sequence is the $d^{\rho(n+1)}$-differential, which is a derivation by Lemma 7.1. This acts upon the $E^{\rho(n+1)}$-term given in Lemma 8.7. We now proceed with the analog of Lemma 7.4, using Lemma 8.5 to generalize formula (7.2).

Lemma 8.10. *In the spectral sequence $\hat{E}^*(C_{2^{n+1}}; \mathbb{Z}/2)$ the classes in the summands*

$$\bigoplus_{k=v_2(i)<n} P_{2(2^k-1)}(te_4) \cdot t^i e_3, \quad \text{and} \quad P_{2(2^n-1)}(te_4)[t^{2^n}, t^{-2^n}] \cdot e_3$$

are permanent cycles. There is a differential

$$d^{\rho(n+1)}(t^{-2^n}) = t^{2^n} \cdot e_3 \cdot (te_4)^{2(2^n-1)}.$$

Proof. The listed classes are infinite cycles since they sit in (even, odd) bidegrees in the image from $\hat{E}^*(S^1)$. They are permanent cycles because their fiber degrees are strictly less than $\rho(n+1) - 1$. In total degree $2^{n+1} - 1$, the classes

$$t^{2^{k+1}-2^n} \cdot e_3 \cdot (te_4)^{2(2^k-1)}$$

for $0 \le k < n$ are easily seen to be permanent cycles of this kind.

(They arise as follows. For each $0 \le k < n$ the differential $d^{\rho(k+1)}(t^{-2^k}) = t^{2^k} \cdot e_3 \cdot (te_4)^{2(2^k-1)}$ implies that $d^{\rho(k+1)}(t^{-2^n})$ is zero, by the formal algebra structure. Thus the nonzero class

$$t^{2^{k+1}-2^n} \cdot e_3 \cdot (te_4)^{2(2^k-1)}$$

in the target bidegree of this zero differential is a permanent cycle.)

These are all distinct, so by Lemma 8.6 these are all the n permanent cycles in this total degree. Hence the next infinite cycle in this total degree, namely $t^{2^n} \cdot e_3 \cdot (te_4)^{2(2^n-1)}$, must be a boundary. Considering the fiber degree of this class, the only possible differential affecting it is the $d^{\rho(n+1)}$-differential given above. □

Lemma 8.11. *In the spectral sequence $\hat{E}^*(S^1; \mathbb{Z}/2)$ the d^r-differentials for $r \le \rho(n+1)$ are determined by being derivations, the formulas $d^{\rho(k+1)}(t^{-2^k}) = t^{2^k} \cdot e_3 \cdot (te_4)^{2(2^k-1)}$ for $0 \le k \le n$, and that e_3 and te_4 are permanent cycles.*

Hence $\hat{E}^*(C_{2^{n+1}}; \mathbb{Z}/2)$ is an algebra spectral sequence through the (E^r, d^r)-term with $r = \rho(n+1)$, with differentials determined by $d^{\rho(k+1)}(t^{-2^k}) \ne 0$ as above, $d^{\rho(n+1)}(u_n) = 0$, and that e_3 and te_4 are permanent cycles.

Proof. The $d^{\rho(n+1)}$-differential just found translates to the S^1-spectral sequence by the Frobenius map, and generates the remaining $d^{\rho(n+1)}$-differentials there by the algebra structure and the permanent cycles e_3 and te_4. Translating back by the Frobenius map again determines the even columns of $\hat{E}^*(C_{2^{n+1}}; \mathbb{Z}/2)$ up to the $d^{\rho(n+1)+1}$-differential, and similarly by the Verschiebung maps from the left shifted S^1-spectral sequence to the odd columns. □

Again, the last step of the induction is to determine the $d^{\rho(n+1)+1}$-differentials. These are indeed nonzero, and so in fact $r_0(n+1) = \rho(n+1) + 1$.

Lemma 8.12. *In the spectral sequence $\hat{E}^*(C_{2^{n+1}}; \mathbb{Z}/2)$ there are differentials*

$$d^{\rho(n+1)+1}(t^{2^{n+1}i} \cdot t^{-2^{n+1}} u_{n+1} e_3) = t^{2^{n+1}i} \cdot e_3 (te_4)^{2^{n+1}-1},$$
$$d^{\rho(n+1)+1}(t^{2^{n+1}i} \cdot t^{-2^{n+1}+1} u_{n+1} e_3 e_4) = t^{2^{n+1}i} \cdot e_3 (te_4)^{2^{n+1}}$$

for all integers i.

Proof. We focus on the cases with $i=0$. The same argument will work for any i when all the classes involved are multiplied by $t^{2^{n+1}i}$. We omit this factor throughout for brevity.

Consider the classes $e_3(te_4)^{2^{n+1}-1}$ and $e_3(te_4)^{2^{n+1}}$ in $\hat{E}^*(C_{2^{n+1}};\mathbb{Z}/2)$. They are in (even, odd) bidegrees, hence infinite cycles. They are in total degrees $2^{n+2}+1$ and $2^{n+2}+3$, where by Lemma 8.6 the abutment has n permanent cycles. By Lemma 8.10, the classes

$$t^{2^{k+1}-2^{n+1}} \cdot e_3 \cdot (te_4)^{2^{k+1}-1} \quad \text{and} \quad t^{2^{k+1}-2^{n+1}} \cdot e_3 \cdot (te_4)^{2^{k+1}}$$

for $0 \le k < n$ are all the permanent cycles in these total degrees. Hence the two infinite cycles considered must be boundaries, and given their fiber degrees (which are $\rho(n+1)+3$ and $\rho(n+1)+7$) it is easy to see that the listed differentials are the only ones which can affect the two classes. There can be no slightly longer differentials from near the horizontal axis, because the candidate classes for supporting such differentials die after the E^4- or the E^{12}-term. □

Lemma 8.13. *In $\hat{E}^*(C_{2^{n+1}};\mathbb{Z}/2)$ the $d^{\rho(n+1)+1}$-differentials are determined by being derivations, the formula*

$$d^{\rho(n+1)+1}(t^{-2^{n+1}}u_{n+1}) = (te_4)^{2^{n+1}-1},$$

and that e_3, te_4 and $t^{\pm 2^{n+1}}$ are $d^{\rho(n+1)+1}$-cycles.

Proof. By the formal action of e_3 the previous lemma gives the differentials

$$d^{\rho(n+1)+1}(t^{2^{n+1}i} \cdot t^{-2^{n+1}}u_{n+1}) = t^{2^{n+1}i} \cdot (te_4)^{2^{n+1}-1},$$
$$d^{\rho(n+1)+1}(t^{2^{n+1}i} \cdot t^{-2^{n+1}+1}u_{n+1}e_4) = t^{2^{n+1}i} \cdot (te_4)^{2^{n+1}}$$

for all integers i. By the formal action of e_3 and $(te_4)^2$ from Lemma 7.3 this determines that all the possible $d^{\rho(n+1)+1}$-differentials of $\hat{E}^*(C_{2^{n+1}};\mathbb{Z}/2)$ are indeed nonzero. These differentials are easily seen to act as derivations with respect to the formal algebra structure. □

Theorem 8.14. *Hypothesis* (8.1) *holds for all $n \ge 1$. Hence each spectral sequence $\hat{E}^*(C_{2^n};\mathbb{Z}/2)$ has E^2-term*

$$\hat{E}^2(C_{2^n};\mathbb{Z}/2) = \mathbb{Z}/2[t,t^{-1},u_n,e_3,e_4]/(u_n^2=0, e_3^2=0),$$

its differentials are derivations and generated by

$$d^{\rho(k+1)}(t^{-2^k}) = t^{2^k} \cdot e_3 \cdot (te_4)^{2(2^k-1)}$$

for all $0 \le k < n$ and

$$d^{\rho(n)+1}(t^{-2^n}u_n) = (te_4)^{2^n-1}$$

while e_3, te_4 and $t^{\pm 2^n}$ are permanent cycles. The first nonzero odd differential has length $r_0(n) = \rho(n)+1$, and the spectral sequence collapses immediately thereafter, *at the $E^{\rho(n)+2}$-term. The E^∞-term is*

$$\hat{E}^\infty(C_{2^n};\mathbb{Z}/2) = P_{2^n-1}(te_4)[t^{2^n}, t^{-2^n}]\{1, e_3\} \oplus \bigoplus_{k=v_2(i)<n} P_{2(2^k-1)}(te_4) \cdot t^i e_3\{1, u_n\}$$

and abuts additively to $\pi_*(\hat{\mathbb{H}}(C_{2^n}, T(\mathbb{Z}));\mathbb{Z}/2)$.

Proof. The case $n=1$ was established as Theorem 4.1. Taken in sequence, Lemmas 8.2–8.13 establish that the hypothesis for a given $n \geq 1$ implies the hypothesis for $n+1$, and so the theorem follows by induction. □

Remark 8.15. We note that integrally

$$\pi_* T(\mathbb{Z})_2^{C_{2^n}} \cong \begin{cases} \hat{\mathbb{Z}}_2^{n+1} & \text{for } * = 0, \\ (\text{finite}) & \text{for } * > 0 \text{ odd}, \\ 0 & \text{otherwise} \end{cases}$$

by a Bockstein argument, since we know that $\pi_* T(\mathbb{Z})^{C_{2^n}}$ is torsion in positive degrees by the norm-restriction sequence. Thus multiplication by η is indeed trivial on the two-torsion in $\pi_* T(\mathbb{Z})^{C_{2^n}}$, in all degrees, and so a posteriori we know that the obstruction to an algebra structure on $E^*(C_{2^n};\mathbb{Z}/2)$ vanishes. One consequence is that $\pi_*(T(\mathbb{Z})^{C_{2^n}};\mathbb{Z}/2)$ has exponent two, not four, in all degrees. None of these results need generalize to the fixed points of $T(A)$ for more general rings A.

9. Spectral sequences related to $TF(\mathbb{Z})_2$

Our next task is to recognize and name the permanent cycles in $E^*(C_{2^n};\mathbb{Z}/2)$ and $\hat{E}^*(C_{2^n};\mathbb{Z}/2)$, abutting to the mod two homotopy of $T(\mathbb{Z})^{hC_{2^n}}$ and $\hat{\mathbb{H}}(C_{2^n}, T(\mathbb{Z}))$, respectively. In the limit over the Frobenius maps the former tends to $E^*(S^1;\mathbb{Z}/2)$, which abuts in nonnegative total degrees to the mod two homotopy of $TF(\mathbb{Z})$, defined as $\mathrm{holim}_F T(\mathbb{Z})^{C_{2^n}}$. We write $TF_*(\mathbb{Z};\mathbb{Z}/2)$ for $\pi_*(TF(\mathbb{Z});\mathbb{Z}/2)$.

Our arguments are related to those of Section 4 of [6], but our bookkeeping proceeds differently, by bringing attention to certain vertical and horizontal bands partitioning the spectral sequences.

Lemma 9.1. *The spectral sequence $E^*(S^1;\mathbb{Z}/2)$ converges to $TF_*(\mathbb{Z};\mathbb{Z}/2)$ in nonnegative total degrees, and has E^∞-term*

$$E^\infty(S^1;\mathbb{Z}/2) = \mathbb{Z}/2[te_4]\{1, e_3\} \oplus \bigoplus_{k=v_2(i)} P_{2(2^{k+1}-1)}(te_4) \cdot e_3 e_4^i$$

in nonnegative fiber degrees.

Proof. This is simply the limiting case of Lemma 8.4 as n tends to infinity. □

Definition 9.2. Let $x_{2r}=(te_4)^r$ be the (only) permanent cycle in $E^*(S^1;\mathbb{Z}/2)$ in total degree $2r$, for $r\geq 0$. Let $\{x_{2r-1}(v)\}_{v\geq 0}$ be the permanent cycles in $E^*(S^1;\mathbb{Z}/2)$ in total degree $2r-1$, for $r\geq 1$, indexed sequentially by decreasing filtration.

For example $x_{2r-1}(0)$ is the permanent cycle closest to the vertical axis, in filtration 0 or -2 for r even and odd, respectively. As v grows, the $x_{2r-1}(v)$ move up and left along a ray within the same total degree. Thus

$$E^\infty(S^1;\mathbb{Z}/2)=\begin{cases}\mathbb{Z}/2\{x_{2r}\} & \text{in total degree } 2r\geq 0,\\ \bigoplus_{v=0}^\infty \mathbb{Z}/2\{x_{2r-1}(v)\} & \text{in total degree } 2r-1.\end{cases}$$

Lemma 9.3. *The permanent cycle $x_{2r-1}(v)$ has even filtration degree s uniquely determined by the inequalities*

(i) $\rho(v)<1-s<\rho(v+1)$

and the congruence

(ii) $-s\equiv 2(r-2) \bmod 2^{v+2}$.

Hence the E^∞-term of $E^*(S^1;\mathbb{Z}/2)$ can be divided into countably many vertical bands separated by vertical lines at the filtration degrees $1-\rho(v)$ for $v\geq 0$, and in each odd total degree there is precisely one permanent cycle in each vertical band, with $x_{2r-1}(v)$ in the vth vertical band (counting from 0).

Proof. Fix a total degree $2r-1$ and consider the vth band. We seek a class $(te_4)^j e_3 e_4^i$ with (a) $2j+3+4i=2r-1$ and (b) $\rho(v)<1+2j<\rho(v+1)$, such that (c) $j<2(2^{k+1}-1)$ where $k=v_2(i)$. Then we have found a permanent cycle in the correct band and total degree. (If $i=0$ condition (c) is void.) The first two conditions can be rewritten as $2i+j=r-2$ and $2(2^v-1)\leq j<2(2^{v+1}-1)$. The latter condition lets us select a unique j so that $j\equiv r-2 \bmod 2^{v+1}$. Then the former condition determines i with $2i\equiv 0 \bmod 2^{v+1}$, so $k=v_2(i)\geq v$. Hence $j<2(2^{v+1}-1)\leq 2(2^{k+1}-1)$ as required by (c).

The selected class is unique. For suppose $(te_4)^{j'} e_3 e_4^{i'}$ also satisfies the three conditions (a), (b) and (c). Then $2i+j=2i'+j'$ and $|j-j'|<2^{v+1}$ so $|i-i'|<2^v$. Since $k=v_2(i)\geq v$ we must have $k'=v_2(i)<v$ unless $i=i'$. Then $2(2^v-1)\leq j'$ and $j'<2(2^{k'+1}-1)$ leads to a contradiction. On the other hand, if $i=i'$, then also $j=j'$, and so the classes are equal. □

Lemma 9.4. *For $r\geq 0$*

$$\pi_{2r}(T(\mathbb{Z})^{C_{2^n}};\mathbb{Z}/2)=\begin{cases}(\mathbb{Z}/2)^n & \text{when } v_2(r)\leq n,\\ (\mathbb{Z}/2)^{n+1} & \text{when } v_2(r)>n.\end{cases}$$

For $r\geq 1$

$$\pi_{2r-1}(T(\mathbb{Z})^{C_{2^n}};\mathbb{Z}/2)=\begin{cases}(\mathbb{Z}/2)^n & \text{when } v_2(r)\leq n,\\ (\mathbb{Z}/2)^{n+1} & \text{when } v_2(r)>n.\end{cases}$$

In the latter cases, the associated graded module has additive generators

$$E^0\pi_{2r-1}(T(\mathbb{Z})^{C_{2^n}};\mathbb{Z}/2)=\begin{cases}\mathbb{Z}/2\{x_{2r-1}(0),\dots,x_{2r-1}(n-1)\} & \text{when } v_2(r)\leq n,\\ \mathbb{Z}/2\{x_{2r-1}(0),\dots,x_{2r-1}(n)\} & \text{when } v_2(r)>n.\end{cases}$$

Proof. The orders of these groups were given in Lemma 8.5. By Remark 8.15 they have exponent two, hence are $\mathbb{Z}/2$-vector spaces. Later we will only use the orders of these groups and not their precise group structure, so we omit the proof of this remark.

To identify the generators in total degree $2r-1$, consider the E^∞-term of $E^*(C_{2^n};\mathbb{Z}/2)$ from Lemma 8.4. The permanent cycles include all the permanent cycles of $E^*(S^1;\mathbb{Z}/2)$ in the filtrations $-s$ with $s \le \rho(n)-2$, i.e., in the vertical bands 0 through $n-1$. These are the classes $x_{2r-1}(v)$ with $0 \le v < n$. In addition there are the exceptional classes

$$(te_4)^{2(2^n-1)} e_3 e_4^i$$

with $v_2(i) \ge n$, which are the $x_{2r-1}(n)$. □

Next we make the same calculations for the Tate spectral sequences.

Lemma 9.5. *The spectral sequence $\hat{E}^*(S^1;\mathbb{Z}/2)$ converges to $TF_*(\mathbb{Z};\mathbb{Z}/2)$ in nonnegative total degrees, and has E^∞-term*

$$\hat{E}^\infty(S^1;\mathbb{Z}/2) = \mathbb{Z}/2[te_4]\{1, e_3\} \oplus \bigoplus_{k=v_2(i)} P_{2(2^k-1)}(te_4) \cdot t^i e_3.$$

Proof. This is the limiting case of Lemma 8.2. □

Definition 9.6. Let $y_{2r} = (te_4)^r$ be the (only) permanent cycle in $\hat{E}^*(S^1;\mathbb{Z}/2)$ in total degree $2r$, for $r \ge 0$. Let $\{y_{2r-1}(v)\}_{v \ge 0}$ be the permanent cycles in $\hat{E}^*(S^1;\mathbb{Z}/2)$ in total degree $2r-1$, indexed sequentially by decreasing filtration.

For example $y_{2r-1}(0)$ is the permanent cycle closest to the horizontal axis, in fiber degree 3 or 7 for r even and odd, respectively. As v grows, the $y_{2r-1}(v)$ move up and left along a ray within the same total degree. Thus

$$\hat{E}^\infty(S^1;\mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2\{y_{2r}\} & \text{in total degree } 2r \ge 0, \\ \bigoplus_{v=0}^\infty \mathbb{Z}/2\{y_{2r-1}(v)\} & \text{in total degree } 2r-1. \end{cases}$$

Lemma 9.7. *The permanent cycle $y_{2r-1}(v)$ has fiber degree $4\tau+3$ with $\tau \ge 0$ uniquely determined by the inequalities*

(i) $\rho(v) < 2\tau + 1 < \rho(v+1)$

and the congruence

(ii) $\tau \equiv r-2 \mod 2^{v+1}$.

Hence the E^∞-term of $\hat{E}^*(S^1;\mathbb{Z}/2)$ can be divided into countably many horizontal bands separated by horizontal lines at (near) the fiber degrees $2\rho(v)+1$ for $v \ge 0$, and in each odd total degree there is precisely one permanent cycle in each horizontal band, with $y_{2r-1}(v)$ in the vth horizontal band.

Proof. The proof is similar to that of Lemma 9.3, and will be omitted. □

Lemma 9.8. *In even total degrees*

$$\pi_{2r}(\hat{\mathbb{H}}(C_{2^n}, T(\mathbb{Z})); \mathbb{Z}/2) = \begin{cases} (\mathbb{Z}/2)^{n-1} & \text{when } v_2(r) < n, \\ (\mathbb{Z}/2)^n & \text{when } v_2(r) \geq n. \end{cases}$$

Likewise, in odd total degrees

$$\pi_{2r-1}(\hat{\mathbb{H}}(C_{2^n}, T(\mathbb{Z})); \mathbb{Z}/2) = \begin{cases} (\mathbb{Z}/2)^{n-1} & \text{when } v_2(r) < n, \\ (\mathbb{Z}/2)^n & \text{when } v_2(r) \geq n. \end{cases}$$

Here the associated graded module has additive generators

$$E^0\pi_{2r-1}(\hat{\mathbb{H}}(C_{2^n}), T(\mathbb{Z})); \mathbb{Z}/2)$$

$$= \begin{cases} \mathbb{Z}/2\{y_{2r-1}(0), \ldots, y_{2r-1}(n-2)\} & \text{when } v_2(r) < n, \\ \mathbb{Z}/2\{y_{2r-1}(0), \ldots, y_{2r-1}(n-1)\} & \text{when } v_2(r) \geq n. \end{cases}$$

Proof. This time the proof is similar to that of Lemma 9.4, and will be omitted. □

Definition 9.9. The line containing the permanent cycles $(te_4)^j e_3$ will be called the *separating line*. Let $[x]$ denote the greatest integer less than or equal to a real number x, and let $\log_2(x)$ be the base two logarithm of x.

Lemma 9.10. *In $E^*(S^1; \mathbb{Z}/2)$ the permanent cycle $(te_4)^{r-2}e_3$ in total degree $2r-1$ (on the separating line) has the name $x_{2r-1}(e)$ where $e = [\log_2(r) - 1]$. In $\hat{E}^*(S^1; \mathbb{Z}/2)$ the same permanent cycle has the name $y_{2r-1}(e)$ for the same e. Hence in total degree $2r-1$ the classes $x_{2r-1}(v)$ and $y_{2r-1}(v)$ with $v < e$ lie above (to the right of) the separating line, the classes $x_{2r-1}(e) = y_{2r-1}(e)$ lie on the separating line, and the classes $x_{2r-1}(v)$ and $y_{2r-1}(v)$ with $v > e$ lie below (to the left of) the separating line.*

Proof. By Lemma 9.3 the class $e_3(te_4)^{r-2}$ is in the vth vertical band of $E^*(S^1; \mathbb{Z}/2)$ where $\rho(v) < 1 - s < \rho(v+1)$ and $s = -2(r-2)$, so $4(2^v - 1) < 2r - 3 < 4(2^{v+1} - 1)$ or equivalently $2^{v+1} \leq r < 2^{v+2}$. Hence $v = [\log_2(r) - 1] = e$. By Lemma 9.7 the same class $e_3(te_4)^{r-2}$ is in the wth horizontal band of $\hat{E}^*(S^1; \mathbb{Z}/2)$ where $\rho(w) < 2\tau + 1 < \rho(w+1)$ and $\tau = r - 2$, so again $4(2^w - 1) < 2r - 3 < 4(2^{w+1} - 1)$ and $w = [\log_2(r) - 1] = e$. □

Thus classes on the separating line have the same x- and y-indices. We reserve the notation $x_{2r-1}(e) = y_{2r-1}(e)$ for the class $x_{2r-1}(v) = y_{2r-1}(v)$ with $v = e = [\log_2(r) - 1]$.

10. The restriction map on $TF(\mathbb{Z})_2$

Our final task is to compute the restriction map $R: TF(\mathbb{Z})_2 \to TF(\mathbb{Z})_2$, and the difference between it and the identity map, in mod two homotopy. This allows us to compute the mod two homotopy of the topological cyclic homology $TC(\mathbb{Z})_2$ of the integers at two, defined in [5] by the fiber sequence

$$TC(\mathbb{Z})_2 \xrightarrow{\pi} TF(\mathbb{Z})_2 \xrightarrow{R-1} TF(\mathbb{Z})_2.$$

By viewing R above as the limit over n of the homotopy restriction maps

$$R_n^h : T(\mathbb{Z})^{C_{2^n}} \to \hat{\mathbb{H}}(C_{2^n}, T(\mathbb{Z})),$$

we make this calculation by determining the induced maps of E^∞-terms from $E^\infty(C_{2^n}; \mathbb{Z}/2)$ to $\hat{E}^\infty(C_{2^n}; \mathbb{Z}/2)$ in terms of the permanent cycles $x_{2r-1}(v)$ and $y_{2r-1}(v)$ from the previous section, and forming products in the homotopy of $TC(\mathbb{Z})_2$ to construct good representatives in $TF_*(\mathbb{Z}; \mathbb{Z}/2)$ for these permanent cycles.

Our arguments largely follow those of Section 7 of [6].

In this section all spectra will be implicitly completed at two. Recall the diagram of fiber sequences from [6], see Proposition 5.1 of [11]:

$$\begin{array}{ccccc} T(\mathbb{Z})_{hC_{2^n}} & \xrightarrow{N_n} & T(\mathbb{Z})^{C_{2^n}} & \xrightarrow{R_n} & T(\mathbb{Z})^{C_{2^{n-1}}} \\ \| & & \downarrow{\scriptstyle \Gamma_n} & & \downarrow{\scriptstyle \hat{\Gamma}_n} \\ T(\mathbb{Z})_{hC_{2^n}} & \xrightarrow{N_n^h} & T(\mathbb{Z})^{hC_{2^n}} & \xrightarrow{R_n^h} & \hat{\mathbb{H}}(C_{2^n}, T(\mathbb{Z})) \end{array} \tag{10.1}$$

The vertical maps induce isomorphisms in mod two homotopy in nonnegative degrees by Theorems 0.2 and 0.3. In particular $\hat{\Gamma}_n$ induces an isomorphism between the groups with associated graded groups computed in Lemmas 9.4 and 9.8. The diagram above is compatible with the Frobenius maps, and in the homotopy limit over n, we get the following diagram of fiber sequences:

$$\begin{array}{ccccc} \Sigma T(\mathbb{Z})_{hS^1} & \xrightarrow{N} & TF(\mathbb{Z}) & \xrightarrow{R} & TF(\mathbb{Z}) \\ \| & & \downarrow{\scriptstyle \Gamma} & & \downarrow{\scriptstyle \hat{\Gamma}} \\ \Sigma T(\mathbb{Z})_{hS^1} & \xrightarrow{N^h} & T(\mathbb{Z})^{hS^1} & \xrightarrow{R^h} & \hat{\mathbb{H}}(S^1, T(\mathbb{Z})). \end{array} \tag{10.2}$$

We wish to compute the map $R : TF(\mathbb{Z}) \to TF(\mathbb{Z})$ in mod two homotopy. By these diagrams we may as well study $R_n^h : T(\mathbb{Z})^{hC_{2^n}} \to \hat{\mathbb{H}}(C_{2^n}, T(\mathbb{Z}))$ for n large, or as an approximation, the homomorphism of E^∞-terms induced by

$$E^* R_n^h : E^*(C_{2^n}; \mathbb{Z}/2) \to \hat{E}^*(C_{2^n}; \mathbb{Z}/2).$$

Definition 10.3. The *topological cyclic homology* of $\mathbb{Z}$ at two is defined by the fiber sequence

$$TC(\mathbb{Z})_2 \xrightarrow{\pi} TF(\mathbb{Z})_2 \xrightarrow{R-1} TF(\mathbb{Z})_2.$$

By Proposition 4.6 of [11] it follows that $TC(\mathbb{Z})_2$ is the underlying space of a homotopy commutative ring spectrum, and that π is a ring map. Let $\partial : \Omega TF(\mathbb{Z})_2 \to TC(\mathbb{Z})_2$ be the induced connecting map. We write $TC_*(\mathbb{Z};\mathbb{Z}/2)$ for $\pi_*(TC(\mathbb{Z});\mathbb{Z}/2)$, and likewise for TF and with $\mathbb{Z}/4$-coefficients. In Theorem 10.9 we will compute the mod two homotopy of $TC(\mathbb{Z})_2$, by determining the map induced by $R-1$.

The *cyclotomic trace map* of [5]

$$\mathrm{trc}\colon K(\hat{\mathbb{Z}}_2)_2 \to TC(\mathbb{Z})_2$$

lifts the circle trace map $K(\hat{\mathbb{Z}}_2)_2 \to TF(\mathbb{Z})_2$ over π, by construction (see [5, Proposition 2.5]). We do not know, and will not use, that trc is a ring map. However, it is an infinite loop space map, and thus induces a $\pi_* Q(S^0)$-module homomorphism on homotopy.

Let $\tilde{v}_4 \in TC_4(\mathbb{Z};\mathbb{Z}/4)$ and $\tilde{\eta}_2 \in TC_2(\mathbb{Z};\mathbb{Z}/2)$ be the images of the classes with the same names in $K_4(\hat{\mathbb{Z}}_2;\mathbb{Z}/4)$ and $\pi_2(Q(S^0);\mathbb{Z}/2)$, respectively. Recall that ρ denotes the mod two reduction map from mod four to mod two homotopy. Define classes

$$\xi_{4i} = \rho((\tilde{v}_4)^i) \in TC_{4i}(\mathbb{Z};\mathbb{Z}/2),$$

$$\xi_{4i+2} = (\tilde{v}_4)^i \cdot \tilde{\eta}_2 \in TC_{4i+2}(\mathbb{Z};\mathbb{Z}/2)$$

for $i \geq 0$. Here the products are formed using the ring structure on the mod four homotopy of $TC(\mathbb{Z})_2$, and its action upon mod two homotopy, stemming from the ring spectrum structure on $TC(\mathbb{Z})_2$. We write $\xi_0 = 1$. Also let

$$\xi_{4i} = \pi(\xi_{4i}) \in TF_{4i}(\mathbb{Z};\mathbb{Z}/2),$$

$$\xi_{4i+2} = \pi(\xi_{4i+2}) \in TF_{4i+2}(\mathbb{Z};\mathbb{Z}/2)$$

be the images in $TF_*(\mathbb{Z};\mathbb{Z}/2)$ of the classes just defined. Then $R(\xi_{2r}) = \xi_{2r}$ in $TF_{2r}(\mathbb{Z};\mathbb{Z}/2)$ for all $r \geq 0$, with R as in (10.2). Also ξ_{2r} is represented (under the identification Γ) by the permanent cycle $x_{2r} = (te_4)^r$ on the separating line in $E^*(S^1;\mathbb{Z}/2)$, and likewise (under the identification $\hat{\Gamma}$) by $y_{2r} = (te_4)^r$ in $\hat{E}^*(S^1;\mathbb{Z}/2)$.

Next let $\lambda \in TC_3(\mathbb{Z})_2$ and $\kappa \in TC_5(\mathbb{Z})_2$ be the images of the classes with the same names in $K_5(\mathbb{Z})_2$. Define classes

$$\xi_{4i+3}(e) = \rho((\tilde{v}_4)^i \cdot \lambda) \in TC_{4i+3}(\mathbb{Z};\mathbb{Z}/2),$$

$$\xi_{4i+5}(f) = \rho((\tilde{v}_4)^i \cdot \kappa) \in TC_{4i+5}(\mathbb{Z};\mathbb{Z}/2)$$

for $i \geq 0$, where $e = [\log_2(2i+2) - 1]$ and $f = [\log_2(2i+3) - 1]$. Again the products are formed using the ring structure on $TC(\mathbb{Z})_2$. Also let

$$\xi_{4i+3}(e) = \pi(\xi_{4i+3}(e)) \in TF_{4i+3}(\mathbb{Z};\mathbb{Z}/2),$$

$$\xi_{4i+5}(f) = \pi(\xi_{4i+5}(f)) \in TF_{4i+5}(\mathbb{Z};\mathbb{Z}/2)$$

be the images in $TF_*(\mathbb{Z};\mathbb{Z}/2)$ of the classes just defined. Then $R(\xi_{2r-1}(e)) = \xi_{2r-1}(e)$ in $TF_{2r-1}(\mathbb{Z};\mathbb{Z}/2)$ for all $r \geq 2$, with $e = [\log_2(r) - 1]$, and $\xi_{2r-1}(e)$ is represented by the permanent cycle $x_{2r-1}(e)=e_3(te_4)^{r-2}$ in $E^*(S^1;\mathbb{Z}/2)$, and likewise by $y_{2r-1}(e)=e_3(te_4)^{r-2}$ in $\hat{E}^*(S^1;\mathbb{Z}/2)$.

Remark 10.4. In Theorem 10.6 we will define classes $\xi_{2r-1}(v)$ in $TF_{2r-1}(\mathbb{Z};\mathbb{Z}/2)$ for arbitrary indices v, not just $v=e=[\log_2(r)-1]$, so as to represent $x_{2r-1}(v)$ in the associated graded group $E^\infty(S^1;\mathbb{Z}/2)$. Then, in Definition 10.8 we will use these $\xi_{2r-1}(v)$ to identify classes ξ_{2r-1} in $TF_{2r-1}(\mathbb{Z};\mathbb{Z}/2)$ that are in the kernel of $R-1$, and hence admit lifts ξ_{2r-1} in $TC_{2r-1}(\mathbb{Z};\mathbb{Z}/2)$. In Theorem 10.9 we describe $TC_*(\mathbb{Z};\mathbb{Z}/2)$ in terms of these classes.

Now consider the homotopy restriction map R_n^h: $T(\mathbb{Z})^{hC_{2^n}} \to \hat{\mathbb{H}}(C_{2^n}, T(\mathbb{Z}))$ and its induced map between the spectral sequences $E^*(C_{2^n};\mathbb{Z}/2)$ and $\hat{E}^*(C_{2^n};\mathbb{Z}/2)$, in terms of the permanent cycles named $x_{2r-1}(v)$ and $y_{2r-1}(v)$. In even total degrees the only permanent cycles are x_{2r} and y_{2r}, respectively, representing $\xi_{2r}\in TF_{2r}(\mathbb{Z};\mathbb{Z}/2)$ in both cases. As noted in Definition 10.3 the restriction map takes x_{2r} to y_{2r}. In odd total degrees there are many more permanent cycles. As we are interested in the limiting case when n grows, we may assume that n is large compared to the degrees of the classes in question.

Proposition 10.5. *Let $r>0$ and $e=[\log_2(r)-1]$. In total degree $2r-1$ the homomorphism*

$$E^\infty R_n^h : E^\infty(C_{2^n};\mathbb{Z}/2) \to \hat{E}^\infty(C_{2^n};\mathbb{Z}/2)$$

maps the permanent cycles $x_{2r-1}(v)$ as follows, provided n is sufficiently large with respect to v:

(1) *$x_{2r-1}(v)$ for $0\le v<e$ maps to 0 or to $y_{2r-1}(w)$, with $v<w<e$,*
(2) *$x_{2r-1}(e)$ maps to $y_{2r-1}(e)$,*
(3) *$x_{2r-1}(e+1)$ maps to 0, and*
(4) *$x_{2r-1}(v)$ for $v>e+1$ maps to $y_{2r-1}(v-1)$.*

Proof. We divide into three cases, namely (i) the classes above the separating line, (ii) the classes $e_3(te_4)^j$ directly on the separating line, and (iii) the classes below the separating line.

Case (i): We consider the map $E^\infty R_n^h : E^\infty(C_{2^n};\mathbb{Z}/2)\to\hat{E}^\infty(C_{2^n};\mathbb{Z}/2)$ in odd total degrees and above the separating line, i.e., on classes $(te_4)^j e_3 e_4^i$ with $i>0$. This is a homomorphism

$$\bigoplus_{k=v_2(i)} \begin{cases} P_{2(2^{k+1}-1)}(te_4)\cdot e_3 e_4^i & \text{if } k<n,\\ P_{2^{n+1}-1}(te_4)\cdot e_3 e_4^i & \text{if } k\ge n \end{cases}$$

$$\xrightarrow{E^\infty R_n^h} \bigoplus_{k=v_2(i)} \begin{cases} P_{2(2^k-1)}(te_4)\cdot e_3 t^{-i} & \text{if } k<n,\\ P_{2^n-1}(te_4)\cdot e_3 t^{-i} & \text{if } k\ge n \end{cases}$$

sending each class to a class with the same name, or zero if there is no such class in the target. The part of the target contained in the second quadrant is

$$\bigoplus_{k=v_2(i)} \begin{cases} P_{2(2^k-1)-i}(te_4)\cdot e_3e_4^i & \text{if } k<n, \\ P_{2^n-1-i}(te_4)\cdot e_3e_4^i & \text{if } k\geq n. \end{cases}$$

Since $i>0$ we have $k<\infty$, so for n large $k<n$, and we may concentrate on the top case. Write $i=2^k\cdot\ell$ with ℓ odd. If $\ell\geq 3$ then $2(2^k-1)-i=(2-\ell)2^k-2\leq 0$. Similarly if $i=1$ or 2. Hence in these cases the target has no classes in the second quadrant and $E^\infty R_n^h$ is the zero homomorphism.

The cases $i=2^k>2$ remain. Then $2(2^k-1)-i=2^k-2$ and

$$P_{2^k-2}(te_4)\cdot e_3e_4^{2^k}$$

survives as the image of $E^\infty R_n^h$. The remaining classes in the source of $E^\infty R_n^h$ are hit by differentials crossing the vertical axis in the Tate spectral sequence, i.e., have representatives in the image of the norm map N_n^h. And $\mathrm{im}(N_n^h)=\ker(R_n^h)$.

When $x_{2r-1}(v)=(te_4)^je_3e_4^{2^k}$ survives to a nonzero class in $\mathrm{im}(E^\infty R_n^h)$ it is represented by some $y_{2r-1}(w)$. If so, we claim $v<w$. Both v and w are clearly less than e, since we are considering classes above the separating line. $x_{2r-1}(v)$ has filtration degree $s=-2j$ satisfying (9.3(i)), so

$$\rho(v)<1+2j.$$

Since $j<2^k-2$ in $\mathrm{im}(E^\infty R_n^h)$ we get $v+2<k+1$. Likewise $y_{2r-1}(w)=(te_4)^je_3e_4^{2^k}$ has fiber degree $4\tau+3$ with $\tau=j+2^k$ satisfying (9.7(i)), and so

$$2j+2^{k+1}+1<\rho(w+1).$$

Since $j\geq 0$ we get $k+1<w+3$. Thus $v<w$. This completes the proof of part (1) of the proposition.

Case (ii): On the separating line we have the classes $e_3(te_4)^{r-2}=x_{2r-1}(e)=y_{2r-1}(e)$. Evidently $E^\infty R_n^h$ is the surjection

$$P_{2^{n+1}-1}(te_4)\cdot e_3 \xrightarrow{E^\infty R_n^h} P_{2^n-1}(te_4)\cdot e_3$$

and so $P_{2^n-1}(te_4)\cdot e_3$ is the image of $E^\infty R_n^h$. As $n\to\infty$ this tends to all of $\mathbb{Z}/2[te_4]\cdot e_3$. In particular

$$E^\infty R_n^h(x_{2r-1}(e))=y_{2r-1}(e).$$

This completes the proof of part (2) of the proposition.

Case (iii): Below the separating line the source of $E^\infty R_n^h$ is the second quadrant part of

$$\bigoplus_{k=v_2(i)} \begin{cases} P_{2(2^{k+1}-1)}(te_4)\cdot e_3e_4^i & \text{if } k<n, \\ P_{2^{n+1}-1}(te_4)\cdot e_3e_4^i & \text{if } k\geq n. \end{cases}$$

Thus $E^\infty R_n^h$ is the homomorphism

$$\bigoplus_{k=v_2(i)} \begin{cases} P_{2(2^{k+1}-1)-i}(te_4)\cdot e_3t^i & \text{if } k<n, \\ P_{2^{n+1}-1-i}(te_4)\cdot e_3t^i & \text{if } k\geq n \end{cases}$$
$$\xrightarrow{E^\infty R_n^h} \bigoplus_{k=v_2(i)} \begin{cases} P_{2(2^k-1)}(te_4)\cdot e_3t^i & \text{if } k<n, \\ P_{2^n-1}(te_4)\cdot e_3t^i & \text{if } k\geq n \end{cases}$$

with $i>0$. Again we may focus on the case $k<n$. Let $i=2^k\cdot\ell$. The highest degree of a class in the target $P_{2(2^k-1)}(te_4)\cdot e_3t^i$ is $4(2^k-1)-2+3-2i=(4-2\ell)2^k-3<0$ if $\ell\geq 3$ or if $i=1$. Otherwise $i=2^k\geq 2$. Then $2(2^{k+1}-1)-i>2(2^k-1)$, and so $E^\infty R_n^h$ is onto in all the cases where the target contains classes in nonnegative degrees. Hence $P_{2(2^k-1)}(te_4)\cdot e_3t^i$ is the image of $E^\infty R_n^h$.

The source of $E^\infty R_n^h$ contains the $2i-1$ classes in odd positive degrees 1 through $4i-3$ from the module $\mathbb{Z}/2[te_4]\cdot e_3t^i$ for $i=2^k\geq 1$, while the target contains only the $i-1$ classes in odd positive degrees 1 through $2i-3$. As i runs through the powers of 2, the lists of odd integers $2i-1, 2i+1, \ldots, 4i-3$ combine to form a list where each odd positive integer appears precisely once. Hence there is precisely one class $(te_4)^je_3t^i$ in each odd positive degree that is contained in the source but not in the target, namely the class of least negative filtration in the source. This is the class in degree $2r-1$ satisfying $2i-3<2r-1\leq 4i-3$, i.e., $x_{2r-1}(e+1)$. The higher-index classes $x_{2r-1}(e+2),\ldots,x_{2r-1}(n)$ and $y_{2r-1}(e+1),\ldots,y_{2r-1}(n-1)$ are present both in the source and the target, simply under these different names. Hence, for n sufficiently large

$$E^\infty R_n^h(x_{2r-1}(v)) = y_{2r-1}(v-1)$$

for $v>e+1$, and

$$E^\infty R_n^h(x_{2r-1}(e+1)) = 0.$$

We may note that $x_{2r-1}(e+1)=d^{4(2^{e+1}-1)}((te_4)^jt^{-2^e})$ for a suitable j, and so this class is represented by a class in $\operatorname{im}(N_n^h)=\ker(R_n^h)$ up to filtration. This completes the proof of parts (3) and (4) of the proposition. □

Now we will choose classes $\xi_{2r-1}^n(v)$ in $\pi_{2r-1}(T(\mathbb{Z})^{C_{2^n}};\mathbb{Z}/2)$ representing $x_{2r-1}(v)$ in $E^\infty(C_{2^n};\mathbb{Z}/2)$, so as to be compatible under the Frobenius maps. Since

$$TF_{2r-1}(\mathbb{Z};\mathbb{Z}/2) = \lim_{F_n} \pi_{2r-1}(T(\mathbb{Z})^{C_{2^n}};\mathbb{Z}/2)$$

the compatible sequence of classes $(\xi_{2r-1}^n(v))_n$ will define a limiting class $\xi_{2r-1}(v) = \lim_{F_n}\xi_{2r-1}^n(v)$ in $TF_{2r-1}(\mathbb{Z};\mathbb{Z}/2)$ representing $x_{2r-1}(v)$ in $E^\infty(S^1;\mathbb{Z}/2)$. (There is no $\lim^1$-term, since the groups involved are finite.) The choices will be made compatibly with Definition 10.3 in the case $v=e$. Proposition 10.5 lifts to give the following two-primary analogue of Theorem 4.2 of [7].

Theorem 10.6. *Let $e=[\log_2(r)-1]$. There are classes $\xi_{2r-1}(v)\in TF_{2r-1}(\mathbb{Z};\mathbb{Z}/2)$ representing $x_{2r-1}(v)$ in the associated graded groups, such that*

(1) *$R(\xi_{2r-1}(v))$ for $0\le v<e$ is a sum of terms $\xi_{2r-1}(w)$ with $v<w<e$,*
(2) *$R(\xi_{2r-1}(e))=\xi_{2r-1}(e)$,*
(3) *$R(\xi_{2r-1}(e+1))=0$, and*
(4) *$R(\xi_{2r-1}(v))=\xi_{2r-1}(v-1)$ for $v>e+1$.*

Proof. There are classes

$$\xi^n_{2r-1}(v)\in\pi_{2r-1}(T(\mathbb{Z})^{C_{2^n}};\mathbb{Z}/2),$$
$$\zeta^n_{2r-1}(v)\in\pi_{2r-1}(\hat{\mathbb{H}}(C_{2^n},T(\mathbb{Z}));\mathbb{Z}/2)$$

representing the classes $x_{2r-1}(v)$ and $y_{2r-1}(v)$ in the respective associated graded groups. Here $0\le v<n$ if $v_2(r)\le n$ and $0\le v\le n$ if $v_2(r)>n$ in the case of the $\xi^n_{2r-1}(v)$, by Lemma 9.4, while $0\le v<n-1$ if $v_2(r)\le n-1$ and $0\le v\le n-1$ if $v_2(r)>n-1$ in the case of the $\zeta^n_{2r-1}(v)$, by Lemma 9.8. The classes for varying n can inductively be chosen to be compatible under the F_n-maps, and thus define classes $\xi_{2r-1}(v)=\lim_{F_n}\xi^n_{2r-1}(v)\in TF_{2r-1}(\mathbb{Z};\mathbb{Z}/2)$. We need to determine the map R on these limiting classes. There is a choice in choosing a lift $\xi^n_{2r-1}(v)$ when given $\xi^{n-1}_{2r-1}(v)$ and $x_{2r-1}(v)\in E^\infty(C_{2^n};\mathbb{Z}/2)$, and we shall make use of this freedom to make "good" choices which simplify the formulas below.

First, we have chosen $\xi_{2r-1}(e)$ in Definition 10.3, and let $\xi^n_{2r-1}(e)$ be its image under the homomorphism induced by the natural map $TF(\mathbb{Z})_2\to T(\mathbb{Z})^{C_{2^n}}$. Then $F_n(\xi^n_{2r-1}(e))=\xi^{n-1}_{2r-1}(e)$ for all n, and since by Proposition 2.5 of [5] the maps R_n and F_n are homotopic on the image from K-theory, also $R_n(\xi^n_{2r-1}(e))=\xi^{n-1}_{2r-1}(e)$.

Second, $x_{2r-1}(e+1)$ is represented by a class x in $E^*(S^1;\mathbb{Z}/2)$ hit by a differential crossing the vertical axis in $\hat{E}^*(S^1;\mathbb{Z}/2)$, say $d^i(y)=x$. Then a representative for y in $\pi_{2r-1}(\Sigma T(\mathbb{Z})_{hS^1};\mathbb{Z}/2)$ maps under N to a class in $TF_{2r-1}(\mathbb{Z};\mathbb{Z}/2)$ representing $x_{2r-1}(e+1)$, and which lies in $\ker(R)$. We choose $\xi_{2r-1}(e+1)\in TF_{2r-1}(\mathbb{Z};\mathbb{Z}/2)$ as the image under N of this representative. So $R(\xi_{2r-1}(e+1))=0$ and similarly $R_n(\xi^n_{2r-1}(e+1))=0$ for all n.

Since $\hat{\Gamma}_n$ is an isomorphism, and compatible with the various F-maps, it follows as in (7.7) and (7.9) of [6], or equivalently (3.20) of [7], that $\xi^{n-1}_{2r-1}(v)$ corresponds under $\hat{\Gamma}_n$ to $\zeta^n_{2r-1}(v)$ modulo terms $\zeta^n_{2r-1}(w)$ with $v<w$. (The conversion matrix from the ξ-basis to the ζ-basis is lower triangular and invertible, hence has 1's on the diagonal.) This pattern persists for all n, and thus also in the limit.

Hence Proposition 10.5 asserts that under the two-adic connective equivalences Γ_n and $\hat{\Gamma}_n$, R^h_n corresponds to the homomorphism

$$\pi_{2r-1}(T(\mathbb{Z})^{C_{2^n}};\mathbb{Z}/2)\xrightarrow{R_n}\pi_{2r-1}(T(\mathbb{Z})^{C_{2^{n-1}}};\mathbb{Z}/2)$$

which maps

(1) $\xi^n_{2r-1}(v)$ for $0\le v<e$ to 0 modulo terms $\xi^{n-1}_{2r-1}(w)$ with $v<w\le n-2$,
(2) $\xi^n_{2r-1}(e)$ to $\xi^{n-1}_{2r-1}(e)$,

(3) $\xi^n_{2r-1}(e+1)$ to 0, and
(4) $\xi^n_{2r-1}(v)$ for $e+2 \le v \le n-1$ to $\xi^{n-1}_{2r-1}(v-1)$ modulo terms $\xi^{n-1}_{2r-1}(w)$ with $v \le w \le n-2$.

Note that with respect to these bases R_n is represented by a $(n-1)\times n$ matrix. The row corresponding to $\xi^{n-1}_{2r-1}(e)$ has a 1 in the column corresponding to $\xi^n_{2r-1}(e)$, followed by zeros to the right. Similarly each row corresponding to $\xi^{n-1}_{2r-1}(v)$ with $e<v$ has a 1 in the column corresponding to $\xi^n_{2r-1}(v+1)$, followed by zeros to the right. By modifying the choice of the lifts $\xi^n_{2r-1}(v)$ of $\xi^{n-1}_{2r-1}(v)$ over F_n for $v>e+1$, we may assume that the rows corresponding to $\xi^{n-1}_{2r-1}(v)$ for $e \le v$ in the matrix representing R_n consist only of zeros, except for the single 1 in each row just mentioned. This is possible because a lift $\xi^n_{2r-1}(v)$ of $\xi^{n-1}_{2r-1}(v)$ is only determined modulo the "new" class $\xi^n_{2r-1}(n-1)$. (A detailed proof would proceed by induction over n.)

With these choices, the limiting classes $\xi_{2r-1}(v) = \lim_n \xi^n_{2r-1}(v)$ have the properties listed in the theorem. □

Using these classes, we may write:

(10.7) $$TF_{2r-1}(\mathbb{Z};\mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2\{\xi_{2r}\} & \text{for } * = 2r \ge 0, \\ \prod_{v=0}^{\infty} \mathbb{Z}/2\{\xi_{2r-1}(v)\} & \text{for } * = 2r-1 > 0. \end{cases}$$

Note that a product appears, rather than a direct sum, since $TF_{2r-1}(\mathbb{Z};\mathbb{Z}/2)$ is the limit of the groups $\pi_{2r-1}(T(\mathbb{Z})^{C_{2^n}};\mathbb{Z}/2)$.

Definition 10.8. Let $r \ge 1$ and $e = [\log_2(r) - 1]$. Define $\xi_{2r-1} = \lim_v \xi_{2r-1}(v)$, i.e.,

$$\xi_{2r-1} = \prod_{v=e+1}^{\infty} \xi_{2r-1}(v) \in TF_{2r-1}(\mathbb{Z};\mathbb{Z}/2).$$

Theorem 10.9. *There are short exact sequences*

$$0 \to \mathbb{Z}/2\{\xi_{2r}\} \xrightarrow{\partial} TC_{2r-1}(\mathbb{Z};\mathbb{Z}/2) \xrightarrow{\pi} \mathbb{Z}/2\{\xi_{2r-1}(e), \xi_{2r-1}\} \to 0$$

for $r \ge 2$, *and*

$$0 \to \mathbb{Z}/2\{\xi_{2r+1}(e)\} \xrightarrow{\partial} TC_{2r}(\mathbb{Z};\mathbb{Z}/2) \xrightarrow{\pi} \mathbb{Z}/2\{\xi_{2r}\} \to 0$$

for $r \ge 1$, *determining* $TC_*(\mathbb{Z};\mathbb{Z}/2)$ *for* $* \ge 2$ *up to extensions. The same sequences determine* $TC_*(\mathbb{Z};\mathbb{Z}/2)$ *in degrees* $-1 \le * \le 1$, *if the nonexistent classes* $\xi_{-1}(e)$, ξ_{-1} *and* $\xi_1(e)$ *are omitted. Hence*

$$\#TC_*(\mathbb{Z};\mathbb{Z}/2) = \begin{cases} 2 & \textit{for } * = 0 \textit{ or } * = -1, \\ 4 & \textit{for } * \ge 2 \textit{ even or } * = 1, \\ 8 & \textit{for } * \ge 3 \textit{ odd}, \\ 1 & \textit{for } * \le -2. \end{cases}$$

Proof. In degree $2r$ with $r \ge 0$

$$\pi_{2r}(R-1;\mathbb{Z}/2) : TF_{2r}(\mathbb{Z};\mathbb{Z}/2) \to TF_{2r}(\mathbb{Z};\mathbb{Z}/2)$$

is the zero homomorphism. In degree $2r-1$ with $r \geq 1$,

$$\pi_{2r-1}(R-1;\mathbb{Z}/2): TF_{2r-1}(\mathbb{Z};\mathbb{Z}/2) \to TF_{2r-1}(\mathbb{Z};\mathbb{Z}/2)$$

is given by

$$\xi_{2r-1}(v) \mapsto \begin{cases} -\xi_{2r-1}(v) + \dots & \text{for } 0 \leq v < e, \\ 0 & \text{for } v = e, \\ -\xi_{2r-1}(e+1) & \text{for } v = e+1, \text{ and} \\ \xi_{2r-1}(v-1) - \xi_{2r-1}(v) & \text{for } v > e+1. \end{cases}$$

The signs are of little importance, since we are working mod two. The ellipsis refers to possible terms $\xi_{2r-1}(w)$ with $v < w < e$. We have defined $\xi_{2r-1} = \lim_v \xi_{2r-1}(v)$, i.e., $\xi_{2r-1} = \prod_{v=e+1}^{\infty} \xi_{2r-1}(v) \in TF_{2r-1}(\mathbb{Z};\mathbb{Z}/2)$. Then for $r \geq 2$

$$\ker \pi_{2r-1}(R-1;\mathbb{Z}/2) = \mathbb{Z}/2\{\xi_{2r-1}(e), \xi_{2r-1}\},$$
$$\operatorname{cok} \pi_{2r-1}(R-1;\mathbb{Z}/2) = \mathbb{Z}/2\{\xi_{2r-1}(e)\}.$$

In the case $r=1$, the class $\xi_{2r-1}(e)$ does not appear. So $\pi_1(R-1;\mathbb{Z}/2)$ is surjective with kernel $\mathbb{Z}/2\{\xi_1\}$. The theorem follows. □

Definition 10.10. Let $\xi_{2r-1} \in TC_{2r-1}(\mathbb{Z};\mathbb{Z}/2)$ denote a lift of

$$\xi_{2r-1} \in \operatorname{im}(\pi) = \ker(R-1) \subseteq TF_{2r-1}(\mathbb{Z};\mathbb{Z}/2).$$

This class is defined modulo classes in $\operatorname{im}(\partial) = \mathbb{Z}/2\{\partial(\xi_{2r})\}$.

To recover the two-adic homotopy type of $TC(\mathbb{Z})$ from the mod two homotopy type, we shall in [18] need the action of $\tilde{v}_4 \in TC_4(\mathbb{Z};\mathbb{Z}/4)$ on $TC_*(\mathbb{Z};\mathbb{Z}/2)$.

Proposition 10.11. *Multiplication by $\tilde{v}_4$ on $TF_*(\mathbb{Z};\mathbb{Z}/2)$ acts as formal multiplication by $(te_4)^2$ on $E^\infty(S^1;\mathbb{Z}/2)$. The action map is injective on* $\ker \pi_*(R-1;\mathbb{Z}/2)$ *and on* $\operatorname{cok} \pi_*(R-1;\mathbb{Z}/2)$, *and is given by*

$$\tilde{v}_4 \cdot \xi_{2r} = \xi_{2r+4}, \qquad \tilde{v}_4 \cdot \xi_{2r-1}(e) = \xi_{2r+3}(f),$$
$$\tilde{v}_4 \cdot \partial(\xi_{2r}) = \partial(\xi_{2r+4}), \quad \tilde{v}_4 \cdot \partial(\xi_{2r-1}(e)) = \partial(\xi_{2r+3}(f)),$$
$$\tilde{v}_4 \cdot \xi_{2r-1} = \xi_{2r+3}$$

with $e = [\log_2(r) - 1]$ and $f = [\log_2(r+2) - 1]$. Hence $\tilde{v}_4$ acts injectively on $TC_*(\mathbb{Z};\mathbb{Z}/2)$ in all degrees. The classes not in the image of the action are additively generated by $\partial(1)$, 1, ξ_1, $\partial(\xi_2)$, ξ_2, $\partial(\xi_3(0))$, $\xi_3(0)$, ξ_3, $\partial(\xi_5(0))$ and $\xi_5(0)$.

Proof. By their definition as classes in $TC_*(\mathbb{Z}_2;\mathbb{Z}/2)$ the classes ξ_{2r} and $\xi_{2r-1}(e)$ are related as claimed by $\tilde{v}_4$. The connecting map ∂ is a $TC_*(\mathbb{Z};\mathbb{Z}/4)$-module map (see Remark 2.9), so multiplication by $\tilde{v}_4$ commutes with ∂.

It remains to consider the limit classes $\xi_{2r-1} = \lim_v \xi_{2r-1}(v) = \prod_{v=e+1}^{\infty} \xi_{2r-1}(v)$ in $\ker \pi_{2r-1}(R-1;\mathbb{Z}/2)$. In $E^\infty(S^1;\mathbb{Z}/2)$ the class ξ_{2r-1} is represented by the limit of the classes $x_{2r-1}(v)$ for v large. Here $x_{2r-1}(v)=(te_4)^j e_3 t^i$ for some $i=2^k$ and $2r-1=2j+3-2i$. These classes are defined and nonzero as long as $j<2(2^{k+1}-1)-i=3i-2$. Then $(te_4)^2 \cdot x_{2r-1}(v)=(te_4)^{j+2}e_3t^i$ is a nonzero permanent cycle if $j+2<3i-2$. This may not hold for a given v, but by choosing v sufficiently large we may increase i and j by a large amount (the same amount, since their difference is controlled by the relation to r), and so achieve the inequality $j+2<3i-2$. Thus $(te_4)^2 \cdot x_{2r-1}(v)=x_{2r+3}(w)$ for some $w>e$ when v is sufficiently large. This proves that $\tilde{v}_4 \cdot \xi_{2r-1}=\xi_{2r+3}$.

Hence $\tilde{v}_4$ acts injectively on the subgroups and quotient groups of the extensions in Theorem 10.9, and therefore also on the total groups $TC_*(\mathbb{Z};\mathbb{Z}/2)$. □

This completes the spectral sequence analysis needed to understand $TC(\mathbb{Z})$ at two, and with it, the completed algebraic K-theory of the two-adic integers. The remaining arguments will be presented in [18].

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